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6. AUTHOR(S)
Jason L. Speyer

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
University of California, Los Angeles
405 Hilgard Avenue
Los Angeles, CA 90095-1597

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13. ABSTRACT *(Maximum 200 words)*

The objective of our efforts was to extend and apply a new adaptive control technique based on a disturbance attenuation bound. The structure of this new adaptive control scheme is the result of formulating a disturbance attenuation problem for a particular class of nonlinear systems whose solution is obtained without *any* approximation. A global solution is obtained and must be contrasted with much of the nonlinear $\mathcal{H}_\infty$ results which assume that the scheme operates *locally* about some equilibrium condition.

The class of nonlinearities considered is that of a linear system where the coefficient matrix of the control is assumed to be a linear function of an unknown parameter. The work performed on this grant extended this class to include state coefficients matrices linear in the parameter if the associated state that multiplies this term is measured perfectly.

To bring these mathematical abstractions to engineering practice, a significant effort was made to apply this new adaptive control scheme to the development of an adaptive flight control system. We are just beginning to show performance improvements in the time response over that of standard adaptive controllers due to an initial reduction in the control effort associated with those control system parameters that are initially uncertain. This new adaptive controller involves not only the state and parameter estimates, but also the pseudo covariance matrix. This new adaptive scheme does require the determination of the global maxima of a certain function with respect to the uncertain parameters.

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Final Technical Report

ROBUST GAME THEORETIC GUIDANCE AND CONTROL LAWS FOR MISSILE SYSTEMS

Grant No. F49620-92-J-0327

Jason L. Speyer
Mechanical, Aerospace and Nuclear Engineering Department
University of California, Los Angeles
Los Angeles, CA 90095

The objective of our efforts was to extend and apply a new adaptive control technique based on a disturbance attenuation bound. The structure of this new adaptive control scheme is the result of formulating a disturbance attenuation problem for a particular class of nonlinear systems whose solution is obtained without *any* approximation. A global solution is obtained and must be contrasted with much of the nonlinear $\mathcal{H}_\infty$ results which assume that the scheme operates *locally* about some equilibrium condition.

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To bring these mathematical abstractions to engineering practice, a significant effort was made to apply this new adaptive control scheme to the development of an adaptive flight control system for a high angle-of-attack aircraft such as the F-18 HARV (High Angle-of-attack Research Vehicle). We are just beginning to show performance improvements in the time response over that of standard adaptive controllers due to an initial reduction in the control effort associated with those control system parameters that are initially uncertain. This new adaptive controller involves not only the state and parameter estimates, but also the pseudo covariance matrix. This new adaptive scheme does require the determination of the global maxima of a certain function with respect to the uncertain parameters.

Our current results are for a single parameter. As in the F-111 adaptive flight control system, we focus on estimating the moment coefficient due to elevator deflection. Our goals are to extend this to multiple parameters and to eliminate the restriction on perfect information so that all parameters in the system can be included. Our approach to removing this restriction is to use perturbation theory associated with an assumed small measurement uncertainty weighting matrix. Complementary results using perturbation theory are being pursued on our main AFOSR grant (F49620-91-0077).

1 Introduction

Current adaptive control schemes assume certain equivalence. That is, the structure of the adaptive controller assumes a parameter identifier in cascade with a controller such as the linear-quadratic-Gaussian (LQG) controller or multi-step predicted output control. Although the proper approach to adaptive control is based on stochastic control theory, it is untractable. Even the simplest extension of the LQG problem is unmechanizable. This extension involves a linear system where the control coefficient matrix is a linear function of an unknown parameter. If this parameter set is augmented to the original state space, a finite dimensional conditional Gaussian estimator is used to reduce this problem to a full information problem. Nevertheless, the resulting stochastic control problem of minimizing the expected value of a quadratic performance index is subject to a stochastic vector differential equation and a Riccati differential equation. To date this problem remains untractable except for perturbation methods based on small measurement noise spectral densities. This approach is showing significant improvement and is being developed under our current AFOSR grant.

An alternate, but deterministic, approach is formulated by determining a controller which bounds a disturbance attenuation function against all admissible measurement and process disturbances and initial conditions. For the stochastic control problem described above, i.e. the control coefficient is a linear function of a parameter vector, a disturbance attenuation controller can be found via a dynamic programming solution [1]. The problem is shown in [1] to decompose into a controller problem with full information with an associated optimal return function representing the optimal cost into the future and an estimation problem with an associated optimal accumulation function representing the optimal cost due to the control and disturbances from the past. The maximum of the sum of these two functions with respect to the unknown current state produces the worst case state. A sufficiency theorem in [2] requires that for a saddle point controller to exist the worst case state must be a unique global maximum. For this class of disturbance attenuation problem, the global maximum are shown to be nonunique. However, in [1] it is shown that the resulting control strategy is still a saddle point strategy since it is proven that the *control strategy* when there is not a unique global worst case state is unique. An alternate and direct proof is given in [3] where the infinite-time results are also presented.

This powerful result forms the basis of a new approach to adaptive control. In the next section we describe the work performed on this supplemental grant. In the following section we describe the work that would have been performed if the no cost extension had been allowed.

2 An Extension of of the Adaptive Controller Based on Disturbance Attenuation with Application to Aircraft Flight Control

In the disturbance attenuation adaptive controller of [1,3] only the control coefficient matrix is a linear function of an uncertain set of parameters. This new adaptive control technique is extended to include some of the parameters in the state coefficient matrix. To ensure that the estimator remains finite dimensional, only the parameters of the state coefficient matrix which multiplies a state element that is perfectly measured are used in the extension of the new adaptive control law. This generalization allows some reduction in the estimator dimension, but the worst case state is produced now from maximizing the sum of optimal return function and the optimal accumulation function subject to a constraint formed by the perfect measurements. Details of this analysis are given in Appendix A. In the next section, we discuss further generalization to the partial information case by perturbation methods.

This research supplement is motivated by the need to learn how to implement this adaptive controller in important applications such as aircraft flight control systems. To evaluate controller design a high fidelity nonlinear simulation of the F-18 high angle-of-attack research vehicle (HARV) was obtained from NASA Dryden. A parameter-robust game theoretic compensator was designed to track pilot inputs in angle-of-attack, sideslip, and stability-axis roll rate through stick and rudder pedal commands. Zero steady state error in the presence of step inputs is achieved by using a system of error coordinates which also allows for thrust vector commands to fade to zero in steady state. Results of this study are given in Appendix B. This controller and its performance were to be used as a benchmark to compare the results of the adaptive controller.

To begin to understand the implementation issues of this new adaptive controller the longitudinal mode of the F-18 HARV was first controlled. Parameter uncertainty in both the state and control matrices were considered. In particular, stability derivatives that multiply the inertial states were used to augment the state vector to be estimated on-line. Since four unknown parameters were included in the adaptive controller, a search for the global maximum of the sum of the optimal return function and optimal accumulation function with respect to these four parameters was not attempted. Rather a possible local worst case state is obtained near the estimated state. The performance of the new adaptive controller was similar to that of standard adaptive controllers assuming certainty equivalence. It appears that this approach to adaptive control is equivalent to nonlinear $\mathcal{H}_\infty$ controllers which assume local behavior near an equilibrium point.

To obtain dramatic performance, it appeared that the global aspects of the controller was to be explored. The parameter set was reduced to a scalar, the moment coefficient due to elevator deflection. For certain initial conditions remarkable performance is obtained over current adaptive controllers. In regimes where the scalar parameter is quite unknown the control emphasizes thrust vectoring and reduces the elevator deflection to be almost zero. As more information is obtained the controller begins to use more elevator deflection and less thrust vectoring which is eventually faded to zero. This is to be contrasted with standard adaptive controllers in which substantial elevator deflection is used early even though the parameter is the wrong sign. Therefore, the initial response is in the wrong direction. The inherent conservative, but intelligent, performance of the new controller is associated with the two worst case states in which usually only one is a global maximum. In the beginning the global worst case state dictates a conservative policy where the elevator deflection is made small and the thrust vectoring dominates the response. At some time, say t_c, both worst case states produce identical cost. Here, there is a switch from the conservative policy to one similar to that of the standard adaptive controller. Note that the controls do remain continuous, even at t_c. Our current study evaluating the performance of this new adaptive controller for flight control is described in detail in Appendix C.

3 Future Work in the Adaptive Disturbance Attenuation Controller and It's Application

Based upon the encouraging performance of the new adaptive flight controller described in Appendix C, the following research directions are proposed. To generalize the current procedure to include partial information and parameter uncertainty in both the state and control coefficient matrices, a perturbation methodology is suggested. Here, two approaches are possible. Either the measurement uncertainty weighting in the disturbance attenuation function could be assumed small and, therefore, an expansion parameter, or the coefficient of the parameter in the state coefficient matrix could be assumed small and, therefore, an expansion parameter. This second approach in a more general setting is being explored in our main current AFOSR grant. Our current results are given in Appendix D. In either approach our current results [1,3,Appendix A] form the zeroth-order solution. As shown in Appendix D, additional higher-order terms remain finite-dimensional.

Application of this new controller has been shown to produce dramatic and intelligent performance for one parameter. Efficient numerical methods are being sought which not only produce the global maximum, but keeps track of all the local maximum to determine when one maximum value moves past another inducing a change in strategy, but keeping the control continuous.

As shown in Appendix C, the new adaptive controller is a rather cautious controller. When the uncertainty in the coefficient of the elevator deflection is high, emphasis is placed on the thrust vectoring. When the parameter becomes better known, the elevator is used more, and in steady state the thrust vectoring is faded out. This is required due to the deterioration of the thrust vectoring paddles used on the F-18 HARV. The strategy of this new adaptive control law is highly governed by not only the state and parametric estimates, but their associated pseudo error variance. This is unique among implementable adaptive controllers. Therefore, our future efforts are directed to understanding the behavior of this new adaptive technique through examples such as the adaptive flight controller of the F-18 HARV, extensions to multiple uncertain parameters, and extensions to the theory to include constraints on the parameter uncertainties. This last item has the difficulty that we wish to preserve the saddle point structure since the controller is, in a sense, certainly equivalent and therefore, easier to compute on-line. Otherwise, one is left with the difficult problem of finding a minimax controller.

4 Conclusion

This supplement to our AFOSR grant has been important in bring to practice an important new adaptive control law conceived and developed from a theoretical viewpoint under AFOSR sponsorship. Charles Dhillon brings to this research project a strong background and interest in the underlying flight control problem and in the application and extension of this new adaptive control law to aircraft and missile flight control systems. Our progress has not been as rapid as desirable, but we have kept our sight squarely on the need to bring important theory to engineering practice, an activity usually overlooked.

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Appendix A
EXTENSIONS OF ROBUST AND ADAPTIVE CONTROLLERS USING DISTURBANCE ATTENUATION

Extensions of Robust and Adaptive Controllers Using Disturbance Attenuation

C. H. Dillon*and J. L. Speyer†
Mechanical, Aerospace and Nuclear Engineering Department
University of California, Los Angeles, CA 90024
speyer@seas.ucla.edu
Tel: (310) 206-4451 Fax: (310) 206-2302

Abstract

The purpose of this paper is to present extensions to robust and adaptive control design techniques based on disturbance attenuation to a class of systems which have a portion of the state space which is measured perfectly (without additive noise) in addition to a set of measurements which are corrupted by noise. For the robust controller, a system results which is similar to that obtained from a Parameter Robust Game Theoretic Synthesis (PRGTS), with a reduction in the order of the state estimator. In the adaptive case, the techniques presented allow for the extension of the estimator to include plant state coefficients which multiply states which are measured perfectly.

Keywords: Robust Adaptive Control, Disturbance Attenuation, Dynamic Programming.

1 Introduction

Several approaches to controlling dynamic systems with uncertain parameters have been developed in recent years [1, 2, 3, 4].The work presented in this paper provides extensions to two such approaches to control of uncertain systems to a class of systems which contains a combination of noisy and perfect measurements. First, a compensator which is robust to changes in system parameters is developed for this class of systems in a manner similar to [2, 4] which results in a

*Research Assistant
†Professor, Mechanical, Aerospace and Nuclear Engineering Department, UCLA

reduced order robust compensator. Secondly, the work of [1] is extended to this class of systems, which results in a compensator which is adaptive to uncertain parameter multiplying perfectly measured states as well as the controls.

In [4], a controller was developed using a game theoretical approach to the solution of a disturbance attenuation problem which, when combined with the use of an internal feedback loop (IFL) decomposition to represent the uncertainties in the coefficients of the linear system, essentially produced a control which is robust to the parameter variations within the model. By extending these results to the class of systems considered in this paper, the order of the compensator is reduced by considering the perfectly measured states in a manner similar to that developed in [5].

In [1], a similar disturbance attenuation problem was formulated, but with an augmented system which included parameter uncertainties in coefficients multiplying the controls. The disturbance attenuation problem for the augmented system was then solved using a dynamic programming approach, similar to that described in [2] to yield an adaptive compensator structure which included an estimator for determining values of the state and control coefficient parameters, a controller as a function of the state but dependent on the parameter values, and a connection condition which linked the two. In the connection condition, the sum of the optimal return function associated with the control problem and the optimal accumulation function associated with the estimation problem resulting from the decomposition of the dynamic programming solution process is maximized with respect to the current state, i.e. the worst case state. For this class of adaptive control problems, in contrast to the results in [2, 3], the adaptive compensator structure is based on the results of [1] where the restrictive assumption that the worst case state be a singleton is not required.

In both of these approaches [1, 4], it was assumed that all measurements received were corrupted by an external disturbance. In some systems, however, it may arise that a subset of the measurements may be known perfectly (or at least close enough to be modeled as such). One such case could be the dynamics of an airplane, where body angular rates are measured very accurately, but measurements of state variables such as angle-of-attack and sideslip are not nearly as easily or precisely measured.

In such a case, where a portion of the measurements may contain noise while another does not, it is possible to extend the methods described above. In the case of the robust controller, by assuming certain measurements are known exactly, a compensator of reduced order may be designed for the system. In the case of the adaptive approach, an additional set of parameters, those multiplying the states which are measured perfectly, can be included in the augmented state vector and subsequently be estimated on-line. The results presented in this paper will provide detail on these extensions.

2 Dynamic System

The class of system under consideration is a linear system with uncertain coefficients multiplying states and controls and state dynamics which are forced by a random disturbance. Additionally, the class of systems considered has a subset of the state space which is measured perfectly in addition to a set of measurements which are corrupted by noise. This system can be written in the following form:

$$\begin{aligned} \dot{x} &= A(\alpha)x + B(\beta)u + \Gamma w & (1) \\ z_1 &= H_1 x + v & (2) \\ z_2 &= H_2 x & (3) \end{aligned}$$

Where $\alpha \in R^k$ and $\beta \in R^l$ are used to designate uncertain parameters in the state and control coefficient matrices.

Two approaches will be considered in determining a control strategy for this system. The first approach is to design the controller so that it is robust to variations in the parameters represented by α and β by using an internal feedback loop (IFL) decomposition to include the parameter uncertainty in the design process, so that the resulting system is in a form consistent with the approach of [4]. Secondly, a controller is considered which is adaptive to the parameters α and β. In order to pose the problem in a manner which is consistent with the theory for each approach, we can first rewrite the dynamic equations of motion in a form more familiar to each method.

2.1 Dynamic System for Reduced Order Robust Controller

In the case of the robust approach, we can write the system state and control coefficient matrices, A and B, in terms of a nominal plant A_0 and B_0, and parameter dependent perturbation ΔA and ΔB:

$$\dot{x} = (A_0 + \Delta A(\alpha))x + (B_0 + \Delta B(\beta))u + \Gamma w \tag{4}$$

where

$$\begin{aligned} \Delta A &= D L_a(\alpha) E \\ \Delta B &= F L_b(\beta) G \end{aligned}$$

Using an IFL decomposition of ΔA and ΔB, we include the parameter uncertainties as a fictitious disturbance, so that the system which we will be analyzing has the standard form:

$$\dot{x} = A_0 x + B_0 u + \bar{\Gamma}\bar{w} \tag{5}$$

where $\bar{w}$ includes external disturbances as well as the fictitious disturbances introduced by the decomposition of the parameter uncertainties and $\bar{\Gamma}$ reflects how these disturbances are introduced into the system.

2.2 Dynamic System for Adaptive Controller with Some State Coefficients Estimated

In order to apply and extend the theory developed in [1] to the class of systems being considered, we first must write the dynamic equations of motion in a form which is consistent with its development. To do this, we can apply a similarity transformation to the original system, if necessary, to obtain a system which is partitioned into the states which are included in the set of perfect measurements and those which are not. This system can then be written in the form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12}(\alpha) \\ A_{21} & A_{22}(\alpha) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1(\beta) \\ B_2(\beta) \end{bmatrix} u + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} w \tag{6}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} H_3 & H_4 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} v \tag{7}$$

where, the parameter uncertainties in the matrices $A_{12}(\alpha)$, $A_{22}(\alpha)$, $B_1(\beta)$ and $B_2(\beta)$ include parameter uncertainties in the form:

$$A_{12}(\alpha) = (A_{12})_0 + \sum_{j=1}^{k} (A_{12})_j \alpha_j \tag{8}$$

$$A_{22}(\alpha) = (A_{22})_0 + \sum_{j=1}^{k} (A_{22})_j \alpha_j \tag{9}$$

$$B_1(\beta) = (B_1)_0 + \sum_{j=1}^{l} (B_1)_j \beta_j \tag{10}$$

$$B_2(\beta) = (B_2)_0 + \sum_{j=1}^{l} (B_2)_j \beta_j \tag{11}$$

We then can form an augmented state vector, $\xi = \left[\ x_1^T x_2^T \alpha^T \beta^T\ \right]^T$, whose dynamics are defined by the system:

$$\dot{\xi} = A_\xi \xi + B_\xi u + \Gamma_\xi w \tag{12}$$

$$z_1 = H_1 \xi + v \tag{13}$$

$$z_2 = H_2 \xi \tag{14}$$

where

$$A_\xi = \begin{bmatrix} A_{11} & (A_{12})_0 & \{ (A_{12})_1 z_2 & \cdots & (A_{12})_k z_2 \} & \{ (B_1)_1 u & \cdots & (B_1)_l u \} \\ A_{21} & (A_{22})_0 & \{ (A_{22})_1 z_2 & \cdots & (A_{22})_k z_2 \} & \{ (B_2)_1 u & \cdots & (B_2)_l u \} \\ 0 & 0 & & 0 & & & 0 & \\ 0 & 0 & & 0 & & & 0 & \end{bmatrix}$$

and

$$B_\xi = \begin{bmatrix} (B_1)_0 \\ (B_2)_0 \\ 0 \\ 0 \end{bmatrix} \qquad \Gamma_\xi = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{H}_1 = \begin{bmatrix} H_3 & H_4 & 0 & 0 \end{bmatrix} \qquad \bar{H}_2 = \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}$$

In this formulation, the perfect measurements z_2 act as an additional known input to the system. Thus, they could effectively be considered as part of an augmented input vector $\bar{u}$, such that the parameter dependence is then only in the augmented input matrix. This then generalizes the class of systems considered in [1] and is consistent with the theory presented therein.

3 The Disturbance Attenuation Problem

For this system, a disturbance attenuation function is formed as in [1], which is essentially the ratio of norms of performance outputs over disturbance inputs. The problem can be written as:

$$D = \frac{\|y\|^2}{\|w\|^2} \leq \frac{1}{\theta} \quad \theta > 0 \tag{15}$$

where the measures of performance outputs, $\|y\|^2$ and $\|w\|^2$ are defined as

$$\|y\|^2 = \|x(t_f)\|^2_{Q_f} + \int_0^{t_f} \left(\|x\|^2_Q + \|u\|^2_R \right) d\tau \tag{16}$$

$$\|w\|^2 = \|\xi(0)\|^2_{P_0^{-1}} + \int_0^{t_f} \left(\|w\|^2_{W^{-1}} + \|v\|^2_{V^{-1}} \right) d\tau \tag{17}$$

where, in the case of the non-augmented system for the robust controller, ξ is simply x.

This disturbance attenuation function can then be converted to a performance index, given by (18), which gives rise to a differential game problem.

$$J = \frac{1}{2} \left\{ \|y\|^2 - \frac{1}{\theta} \|w\|^2 \right\} \leq 0 \tag{18}$$

The basic idea, then, is to find the control, u, which minimizes the disturbance attenuation function subject to the worst case maximizing disturbance inputs provided by initial conditions, $x(0)$, parameter uncertainty (α, β), state noise w, and measurement noise v.

4 Dynamic Programming Solution

The approach taken in finding the solution to the disturbance attenuation problem is to use a dynamic programming technique to separate the problem into two separate problems. The first, a control subproblem, defines an optimal return function, $\Psi(t,,\xi_t)$ as in [1]. The second, a filtering subproblem, defines an optimal accumulation function, $\Upsilon(\xi_t)$. The dynamic programming problem, then, is to find the values of state and parameters, $\breve{x}, \breve{\alpha}, \breve{\beta}$ such that:

$$\Upsilon(\breve{x}_t, \alpha, \beta) + \Psi(\breve{x}_t, \alpha, \beta) \geq \Upsilon(x_t, \alpha, \beta) + \Psi(x_t, \alpha, \beta) \tag{19}$$

4.1 Application to Reduced Order Robust Controller

In applying this technique to the reduced order robust controller, we note that there is no direct parameter dependence in the performance index and optimal return and accumulation functions. This is due to the fact that, by virtue of the problem formulation, this parameter dependence is essentially hidden in the disturbance input, w.

4.1.1 The Control Subproblem

The control subproblem is formulated as in [1], with a performance index as shown in (20).

$$\begin{aligned} J_c[t, t_f] &= \frac{1}{2}\{\|x(t_f)\|^2_{Q_{t_f}} \\ &+ \int_t^{t_f} [\|x\|^2_Q + \|u\|^2_R - \frac{1}{\theta}(\|w\|^2_{W^{-1}} + \|v\|^2_{V^{-1}})] d\tau\} \end{aligned} \tag{20}$$

Taking $v_t^{t_f} \equiv 0$, as in [1], and adding the zero quantity

$$\frac{1}{2}\int_t^{t_f} \frac{d}{d\tau}[x^T \Pi x] d\tau - \frac{1}{2}[x^T \Pi x]_t^{t_f}$$

to (20), the optimal control and state disturbance and a Riccati differential equation similar to that obtained in [1] are determined by substituting and completing squares.

$$
\begin{aligned}
-\dot{\Pi} &= A^T\Pi + \Pi A + Q - \Pi(BR^{-1}B^T - \theta\Gamma W\Gamma^T)\Pi \\
&\Pi(t_f) = Q_{t_f} && (21) \\
&u_c = -R^{-1}B^T\Pi x && (22) \\
&w_c = \theta W\Gamma^T\Pi x && (23)
\end{aligned}
$$

Substituting these relationships back into the performance index (20), an expression for an optimal return function is obtained. This optimal return function is then given by:

$$
\Psi(x_t) = \frac{1}{2}x_t^T\Pi(t)x_t \tag{24}
$$

4.1.2 The Filtering Subproblem

Next, the filtering subproblem is considered. The approach taken in solving this portion of the problem is essentially the same as that outlined in [5]. Considering only the problem from initial time 0 to current time t, and making use of the measurement equation (2), the performance index for this portion of the problem can be written as:

$$
\begin{aligned}
J_f[0,t] &= \frac{1}{2}\{-\frac{1}{\theta}\|x(0)-\hat{x}_0\|^2_{P_0^{-1}} \\
&+ \int_0^t [\|x\|^2_Q + \|u\|^2_R - \frac{1}{\theta}(\|w\|^2_{W^{-1}} + \|z_1 - H_1x\|^2_{V^{-1}})]d\tau\} \quad (25)
\end{aligned}
$$

Adjoining the dynamics (1) and perfect measurements (3) to the performance index with multipliers λ and $\dot{\mu}$, (25) can be rewritten as:

$$
\begin{aligned}
J_f[0,t] &= J_f[0,t] \\
&-\frac{1}{\theta}\{\int_0^t [\lambda^T(Ax + Bu + \Gamma w - \dot{x}) + \dot{\mu}^T(z_2 - H_2x)]d\tau\}
\end{aligned}
$$

Then, integrating by parts and solving for first order necessary conditions for an extremum yield the relations:

$$
\begin{aligned}
x(0) &= \hat{x}_0 - P_0(\lambda(0) + H_2^T\mu(0)) && (26) \\
\dot{\lambda} &= -A^T\lambda - (H_1^TV^{-1}H_1 - \theta Q)x - A^TH_2^T\mu + H_1^TV^{-1}z_1 && (27) \\
w_f &= -W\Gamma^T(\lambda + H_2^T\mu) && (28)
\end{aligned}
$$

First, the initial conditions are considered. By combining the relations (26) and $z_2(0) = H_2x(0)$, an expression for $\mu(0)$ is obtained.

$$\mu(0) = -(H_2P_0H_2^T)^{-1}(z_2(0) - H_2\hat{x}_0 + H_2P_0\lambda(0)) \tag{29}$$

Substituting this expression back into (26), we obtain the relationship:

$$\begin{aligned}
x(0+) &= \hat{x}(0+) - P(0+)\lambda(0+) && (30)\\
\hat{x}(0+) &= \hat{x}_0 + P_0H_2^T(H_2P_0H_2^T)^{-1}(z_2(0) - H_2\hat{x}_0) && (31)\\
P(0+) &= P_0 - P_0H_2^T(H_2P_0H_2^T)^{-1}H_2P_0 && (32)
\end{aligned}$$

This represents the state, costate, state estimate, and Riccati solution at the time when the first measurement, $z_2(0)$, is received. An important observation is that

$$H_2\hat{x}(0+) = z_2(0)$$
$$H_2P(0+) = 0 \qquad P(0+)H_2^T = 0 \qquad H_2P(0+)H_2^T = 0$$

That is to say, once the first measurement is obtained, that portion of the state contained in the perfect measurements, z_2, is known perfectly. Also, the Riccati solution $P(\cdot)$ becomes singular once measurements are taken.

To obtain an expression for μ for time $t > 0$, we can combine (1) and (28) with the constraint $z_2(\cdot) = H_2x(\cdot)$, which yields the expression:

$$\mu(\cdot) = -(H_2\Gamma W\Gamma^TH_2^T)^{-1}\{\dot{z}_2(\cdot) - H_2[Ax(\cdot) + Bu(\cdot) - \Gamma W\Gamma^T\lambda(\cdot)]\} \tag{33}$$

This relationship, along with (28) can then be substituted back into (1) and (27) to give a system of differential equations for x and λ.

$$\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} &= \begin{bmatrix} \bar{A} & -\Gamma\bar{W}\Gamma^T \\ -(\bar{H}^T\bar{R}^{-1}\bar{H} - \theta Q) & -\bar{A}^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \\
&\quad + \begin{bmatrix} \bar{B} \\ -\bar{H}_2^T\bar{R}_2^{-1}H_2B \end{bmatrix} u + \begin{bmatrix} G\dot{z}_2 \\ -\bar{H}_2^T\bar{R}^{-1}\bar{z} \end{bmatrix}
\end{aligned} \tag{34}$$

where the quantities above are represented by:

$$\bar{R}_2 = H_2\Gamma W\Gamma^TH_2^T \qquad \bar{H}_2 = H_2A \qquad G = \Gamma W\Gamma^TH_2^T\bar{R}_2^{-1}$$
$$\bar{A} = [I - GH_2]A \qquad \bar{B} = [I - GH_2]B \qquad \Gamma\bar{W}\Gamma^T = [I - GH_2]\Gamma W\Gamma^T$$
$$\bar{H} = \begin{bmatrix} H_1 \\ \bar{H}_2 \end{bmatrix} \qquad \bar{R} = \begin{bmatrix} V & 0 \\ 0 & \bar{R}_2 \end{bmatrix} \qquad \bar{z} = \begin{bmatrix} z_1 \\ \dot{z}_2 \end{bmatrix}$$

Next, as in [4], differential equations for the propagation of the state estimate $\hat{x}(\cdot)$ and Riccati solution $P(\cdot)$ are obtained by differentiating the relationship

$$x(t) = \hat{x}(t) - P(t)\lambda(t) \tag{35}$$

This results in the differential equations:

$$\begin{aligned} \dot{\hat{x}} &= (\bar{A} + \theta PQ)\hat{x} + (\bar{B} - P\bar{H}_2^T \bar{R}_2^{-1} H_2 B)u \\ &\quad + P\bar{H}^T \bar{R}^{-1}(\bar{z} - H_1\hat{x}) + G\dot{z}_2 \end{aligned} \tag{36}$$

$$\begin{aligned} \dot{P} &= \bar{A}P + P\bar{A}^T - P(\bar{H}^T\bar{R}^{-1}\bar{H} - \theta Q)P + \Gamma\bar{W}\Gamma^T \\ P(0) &= P_0 \end{aligned} \tag{37}$$

It should be noted that the differential equation which governs the dynamics of the state estimate, $\hat{x}$, contains the derivative of the perfect measurements, $\dot{z}_2$. For the sake of implementation, this is not desirable, but can easily be remedied by defining a transformed state estimate which contains only the measurement itself, z_2 [5]. The estimator can then be implemented by using the transformation

$$\bar{x} = \hat{x} - \left(P\bar{H}2^T \bar{R}_2^{-1} + G\right) z_2 \tag{38}$$

The next step is to determine the optimal accumulation function. To do this, we first substitute the relationships (28) and (33) into (25) and add the zero quantity

$$-\frac{1}{2\theta}\{[\lambda^T P\lambda]_0^t - \int_0^t \frac{d}{d\tau}[\lambda^T P\lambda]d\tau\}$$

Then, the performance index, (25) reduces to

$$\begin{aligned} J_f[0,t] &= -\frac{1}{2\theta}\lambda^T P\lambda + \frac{1}{2}\int_0^t [\|\hat{x}\|_Q^2 + \|u\|_R^2 \\ &\quad -\frac{1}{\theta}\|z_1 - H_1\hat{x}\|_{V^{-1}}^2 - \frac{1}{\theta}\|\dot{z}_2 - H_2(A\hat{x} + Bu)\|_{\bar{R}_2^{-1}}^2]d\tau \end{aligned} \tag{39}$$

Next, we would like to express this performance index in terms of the error between actual and estimated state, $e(\cdot) = x(\cdot) - \hat{x}(\cdot)$. Care must be taken, since $P(\cdot)$ is singular for all $t > 0$. First, we define a transformed error, $\bar{e}(\cdot)$:

$$\bar{e} = \begin{bmatrix} \bar{e}_0 \\ \bar{e}_2 \end{bmatrix} = \begin{bmatrix} H_0 \\ H_2 \end{bmatrix} e \tag{40}$$

where H_0 is defined to span the portion of the state space not contained in the perfect measurements $z_2(\cdot)$. Also, a transformed costate $\bar{\lambda}(\cdot)$ is defined such that

$$\lambda = \begin{bmatrix} H_0 \\ H_2 \end{bmatrix}^T \bar{\lambda} = \begin{bmatrix} H_0 \\ H_2 \end{bmatrix}^T \begin{bmatrix} \bar{\lambda}_0 \\ \bar{\lambda}_2 \end{bmatrix} \tag{41}$$

Then, using these relationships, we can write

$$\begin{aligned} \lambda^T P \lambda &= \bar{e}_0^T (H_0 P H_0^T)^{-1} \bar{e}_0 \\ &= (x-\hat{x})^T H_0^T (H_0 P H_0^T)^{-1} H_0 (x-\hat{x}) \end{aligned} \tag{42}$$

Substituting back into the performance index, we then obtain the desired expression for the optimal accumulation function:

$$\Upsilon(x_t) = g(u, z_1, \dot{z}_2) - \frac{1}{2\theta} e^T H_0^T (H_0 P H_0^T)^{-1} H_0 e \tag{43}$$

where

$$\begin{aligned} \dot{g}(u, z, \dot{z}_2) &= \frac{1}{2}[\hat{x}^T Q \hat{x}] - \frac{1}{2\theta}(z_1 - H_1\hat{x})^T V^{-1}(z_1 - H_1\hat{x}) \\ &\quad - \frac{1}{2\theta}[\dot{z}_2 - H_2(A\hat{x} + Bu)]^T \bar{R}_2^{-1}[\dot{z}_2 - H_2(A\hat{x} + Bu)] \\ g(\cdot,\cdot,\cdot)_{t=0} &= \|z_2 - H_2\hat{x}_0\|^2_{(H_2 P H_2^T)^{-1}} \end{aligned} \tag{44}$$

4.1.3 The Connection Condition

Having determined expressions for the optimal return and optimal accumulation functions, we then must determine the connection conditions which bring together the two solutions. The connection condition, then, is obtained similarly to the method presented in [1], from the constrained relationship

$$\max_{x_t} \left[\Psi(x_t, t) + \Upsilon(x_t, t) + \frac{1}{\theta}\mu(t)^T (z_2(t) - H_2 x_t) \right] \tag{45}$$

which is solved by considering

$$\begin{aligned} &\left(\frac{\partial \Psi}{\partial x_t}\right)^T + \left(\frac{\partial \Upsilon}{\partial x_t}\right)^T - \frac{1}{\theta} H_2^T \mu(t) = 0 \\ &= \Pi x - \frac{1}{\theta} H_0^T (H_0 P H_0^T)^{-1} H_0 (x - \hat{x}) - \frac{1}{\theta} H_2^T \mu(t) \end{aligned} \tag{46}$$

The desired relationship between the optimal state and state estimate, then, is determined by combining the above with the constraint relation $z_2(t) = H_2x(t)$ and solving the matrix equation:

$$\begin{bmatrix} \left(H_0^T \left(H_0P(t)H_0^T\right)^{-1} H_0 - \theta\Pi(t)\right) & H_2^T \\ H_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} H_0^T \left(H_0P(t)H_0^T\right)^{-1} H_0\hat{x} \\ z_2 \end{bmatrix} \quad (47)$$

Combining this relationship with the optimal control (22) and a transformed version of the reduced order optimal state estimate (36), we can then construct a reduced order compensator for the system.

Furthermore, by examining the second variation in the manifold of the constraint $z_2 = H_2x$, we can obtain an expression for a spectral radius condition for the reduced order compensator. This is then given by:

$$P_{z_2}^{\perp}\left(\tilde{P} - \theta\Pi\right)P_{z_2}^{\perp} \geq 0 \quad (48)$$

where

$$\begin{aligned} \tilde{P} &= H_0^T \left(H_0P(t)H_0^T\right)^{-1} H_0 \\ P_{z_2}^{\perp} &= I - H_2^T \left(H_2H_2^T\right)^{-1} H_2 \end{aligned}$$

Note, by the definition of H_0, we have $H_0H_2^T = H_2H_0^T = 0$, so we can rewrite the above as

$$\tilde{P} - \theta P_{z_2}^{\perp}\Pi P_{z_2}^{\perp} \geq 0 \quad (49)$$

4.2 Application to Adaptive Controller with Some State Coefficients Estimated

In applying the dynamic programming technique to the augmented system (12), we first rewrite the performance index to reflect the augmented state vector, as in (50).

$$J = \frac{1}{2}\left\{\|\xi(t_f)\|^2_{\bar{Q}_f} - \frac{1}{\theta}\left\|\xi(0) - \hat{\xi}_0\right\|^2_{\bar{P}_0^{-1}} + \int_0^{t_f}\left[\|\xi\|^2_{\bar{Q}} + \|u\|^2_R - \frac{1}{\theta}\left(\|w\|^2_{W^{-1}} + \|v\|^2_{V^{-1}}\right)\right]d\tau\right\} \quad (50)$$

where, to reflect the extended dynamic system, $\bar{Q}$ and $\bar{Q}_f$ are defined as

$$\bar{Q}_f = \begin{bmatrix} Q_f & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \bar{Q} = \begin{bmatrix} Q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can then break the problem into a separate control and filtering problem, and determine the connection condition which joins the two.

4.2.1 The Control Subproblem

The control subproblem is formulated as in [1], with a performance index as shown in (51).

$$J_c[t,t_f] = \frac{1}{2}\left\{\|x(t_f)\|^2_{Q_f} + \int_t^{t_f}\left[\|x\|^2_Q + \|u\|^2_R - \frac{1}{\theta}\left(\|w\|^2_{W^{-1}} + \|v\|^2_{V^{-1}}\right)\right]d\tau\right\} \quad (51)$$

Taking $v_t^{t_f} \equiv 0$, as in [1], and adding the zero quantity

$$\frac{1}{2}\int_t^{t_f}\frac{d}{d\tau}\left[x^T(\tau)\Pi(\alpha,\beta,\tau)x(\tau)\right]d\tau - \frac{1}{2}\left[x^T(\cdot)\Pi(\alpha,\beta,\cdot)x(\cdot)\right]_t^{t_f}$$

to (51), the optimal control and state disturbance and a Riccati differential equation similar to that obtained in [1] are determined by substituting and completing squares.

$$\begin{aligned} -\dot{\Pi}(\alpha,\beta,\tau) = \quad & A^T\Pi(\alpha,\beta,\tau) + \Pi(\alpha,\beta,\tau)A + Q - \\ & \Pi(\alpha,\beta,\tau)\left(BR^{-1}B^T - \theta\Gamma W\Gamma^T\right)\Pi(\alpha,\beta,\tau) \\ & t < \tau < t_f \\ & \Pi(\alpha,\beta,t_f) = Q_f \end{aligned} \quad (52)$$

$$u(t) = -R^{-1}B^T\Pi(\alpha,\beta,t)x(t) \quad (53)$$

$$w(t) = \theta W\Gamma^T\Pi(\alpha,\beta,t)x(t) \quad (54)$$

Substituting these relationships back into the performance index (51), an expression for an optimal return function is obtained. This optimal return function is then given by:

$$\Psi(x_t,\alpha,\beta,) = \frac{1}{2}x_t^T\Pi(\alpha,\beta,t)x_t \quad (55)$$

4.2.2 The Filtering Subproblem

Next, the filtering subproblem is considered. The approach taken in solving this portion of the problem is essentially the same as that outlined in [5]. Considering only the problem from initial time 0 to current time t, and making use of the measurement equation (7), the performance index for this portion of the problem can be written as:

$$J_f[0,t] = \frac{1}{2}\left\{-\frac{1}{\theta}\left\|\xi(0)-\hat{\xi}_0\right\|^2_{\bar{P}_0^{-1}} + \int_0^t \left[\|\xi\|^2_Q + \|u\|^2_R - \frac{1}{\theta}\left(\|w\|^2_{W^{-1}} + \|z_1 - H_1\xi\|^2_{V^{-1}}\right)\right] d\tau\right\} \quad (56)$$

Adjoining the dynamic constraints to the performance index with multipliers λ and $\dot{\mu}$, J_f can then be written as

$$J_f[0,t] = J_f[0,t] - \frac{1}{\theta}\left\{\int_0^t \left[\lambda^T\left(A_\xi\xi + B_\xi u + \Gamma_\xi w - \dot{\xi}\right) + \dot{\mu}^T(z_2 - H_2\xi)\right] d\tau\right\}$$

Then, integrating the $\dot{\mu}$ terms by parts and solving for first order necessary conditions for an extremum yield the relations:

$$\xi(0) = \hat{\xi}_0 - \bar{P}_0\left(\lambda(0) + H_2^T\mu(0)\right) \quad (57)$$

$$\mu(0) = -\left(H_2\bar{P}_0H_2^T\right)^{-1}\left[z_2(0) - H_2\hat{\xi}_0 + H_2\bar{P}_0\lambda(0)\right] \quad (58)$$

$$\dot{\lambda} = -A_\xi^T\lambda - \left(H_1^TV^{-1}H_1 - \theta\bar{Q}\right)\xi - A_\xi^TH_2^T\mu - H_1^TV^{-1}z_1 \quad (59)$$

$$w = -W\Gamma^T\left(\lambda + H_2^T\mu\right) \quad (60)$$

Combining the constraint $z_2(0) = H_2\xi(0)$ with the relationship obtained for $\mu(0)$, we get an expression for the initial conditions upon receiving the first measurement, $z_2(0)$

$$\xi(0+) = \hat{\xi}(0+) - \bar{P}(0+)\lambda(0+) \quad (61)$$

where

$$\hat{\xi}(0+) = \hat{\xi}_0 + \bar{P}_0H_2^T\left(H_2\bar{P}_0H_2^T\right)^{-1}\left(z_2(0) - H_2\hat{\xi}_0\right) \quad (62)$$

$$\bar{P}(0+) = \bar{P}_0 - \bar{P}_0H_2^T\left(H_2\bar{P}_0H_2^T\right)^{-1}H_2\bar{P}_0 \quad (63)$$

An important result of these relationships is that upon receiving the first "perfect" measurement, the estimate of the perfectly measured states is then simply z_2, and the matrix $\bar{P}$ becomes singular, with null space corresponding to the portion of the state space which is measured perfectly. That is,

$$H_2\hat{\xi}(0+) = z_2(0)$$
$$H_2\bar{P}(0+) = 0 \qquad \bar{P}(0+)H_2^T = 0 \qquad H_2\bar{P}(0+)H_2^T = 0$$

Next, to obtain an expression for μ for time $t > 0$, we consider

$$\dot{z}_2 = H_2\dot{\xi} = H_2\left[A_\xi\xi + B_\xi u - \Gamma_\xi W\Gamma_\xi^T\left(\lambda + H_2^T\mu\right)\right] \tag{64}$$

Then,

$$\mu(\cdot) = -\left(H_2\Gamma_\xi W\Gamma_\xi^T H_2^T\right)^{-1}\left[\dot{z} - H_2\left(A_\xi\xi + B_\xi u - \Gamma_\xi W\Gamma_\xi^T\lambda\right)\right] \tag{65}$$

This then results in a system of differential equations given by

$$\begin{aligned} \begin{bmatrix} \dot{\xi} \\ \dot{\lambda} \end{bmatrix} &= \begin{bmatrix} \bar{A}_\xi & -\Gamma_\xi\bar{W}\Gamma_\xi^T \\ -\left(\bar{H}^T\bar{R}^{-1}\bar{H} - \theta\bar{Q}\right) & -\bar{A}_\xi^T \end{bmatrix}\begin{bmatrix} \xi \\ \lambda \end{bmatrix} \\ &+ \begin{bmatrix} \bar{B}_\xi \\ -\bar{H}_2^T\bar{R}_2^{-1}H_2B_\xi \end{bmatrix}u + \begin{bmatrix} 0 & G \\ H_1^TV^{-1} & \bar{H}_2^T\bar{R}_2^{-1} \end{bmatrix}\begin{bmatrix} z_1 \\ \dot{z}_2 \end{bmatrix} \end{aligned} \tag{66}$$

with the matrices above defined as

$$\begin{array}{lll} \bar{R}_2 = H_2\Gamma_\xi W\Gamma_\xi^T H_2^T & \bar{H}_2 = H_2A_\xi & G = \Gamma_\xi W\Gamma_\xi^T H_2^T\bar{R}_2^{-1} \\ \bar{A}_\xi = [I - GH_2]A_\xi & \bar{B}_\xi = [I - GH_2]B_\xi & \Gamma_\xi\bar{W}\Gamma\xi^T = [I - GH_2]\Gamma_\xi W\Gamma_\xi^T \\ \bar{H} = \begin{bmatrix} H_1 \\ \bar{H}_2 \end{bmatrix} & \bar{R} = \begin{bmatrix} V & 0 \\ 0 & \bar{R}_2 \end{bmatrix} & \end{array}$$

Then, differentiating the relationship

$$\xi(t) = \hat{\xi}(t) - \bar{P}(t)\lambda(t) \tag{67}$$

we obtain differential relations for an estimator and Riccati equation given by

$$\begin{aligned} \dot{\hat{\xi}} &= \left(\bar{A}_\xi + \theta\bar{P}\bar{Q}\right)\hat{\xi} + \left(\bar{B}_\xi - \bar{P}\bar{H}_2^T\bar{R}_2^{-1}H_2B_\xi\right)u \\ &\quad + \bar{P}H_1^TV^{-1}\left(z_1 - H_1\hat{\xi}\right) + \bar{P}\bar{H}_2^T\bar{R}_2^{-1}\left(\dot{z}_2 - \bar{H}_2\hat{\xi}\right) + G\dot{z}_2 \end{aligned} \tag{68}$$

$$\begin{aligned} \dot{\bar{P}} &= \bar{A}_\xi\bar{P} + \bar{P}\bar{A}_\xi^T - \bar{P}\left(\bar{H}^T\bar{R}^{-1}\bar{H} - \theta\bar{Q}\right)\bar{P} + \Gamma_\xi\bar{W}\Gamma\xi^T \\ &\quad \bar{P}(0) = \bar{P}(0+) \end{aligned} \tag{69}$$

As in the case of the robust controller design, it is desirable to introduce a transformation which eliminates the measurement derivative, $\dot{z}_2$ from the estimator equation. This transformation is given for this system by

$$\bar{\xi} = \hat{\xi} - \left(\bar{P}\bar{H}_2^T \bar{R}_2^{-1} + G\right) z_2 \tag{70}$$

The next step is to determine the optimal accumulation function. To do this, we first substitute the relationships obtained above into J_f and add the zero quantity:

$$-\frac{1}{2\theta}\left\{\left[\lambda^T \bar{P} \lambda\right] - \int_0^t \frac{d}{d\tau}\left[\lambda^T \bar{P} \lambda\right] d\tau\right\} \tag{71}$$

Then, we can rewrite J_f as

$$\begin{aligned} J_f[0,t] &= -\frac{1}{2\theta}\lambda^T(t)\bar{P}(t)\lambda(t) + \\ & \frac{1}{2}\int_0^t \left[\left\|\hat{\xi}\right\|_{\bar{Q}}^2 + \|u\|_R^2 - \frac{1}{\theta}\left\|z_1 - H_1\hat{\xi}\right\|_{V^{-1}}^2 - \frac{1}{\theta}\left\|\dot{z}_2 - H_2\left(A_\xi\hat{\xi} + B_\xi u\right)\right\|_{\bar{R}_2^{-1}}^2\right] d\tau \end{aligned}$$

Next, we define the error between augmented state and estimate as $e = \xi - \hat{\xi}$, and define

$$\bar{e} = \begin{bmatrix} \bar{e}_0 \\ \bar{e}_2 \\ \bar{e}_\alpha \\ \bar{e}_\beta \end{bmatrix} = \begin{bmatrix} H_0 \\ H_2 \\ H_\alpha \\ H_\beta \end{bmatrix} e \tag{72}$$

where

$$H_\alpha = \begin{bmatrix} 0 & 0 & I_{k\times k} & 0 \end{bmatrix} \qquad H_\beta = \begin{bmatrix} 0 & 0 & 0 & I_{l\times l} \end{bmatrix}$$

Also, the multiplier λ can be written as

$$\lambda = \begin{bmatrix} H_0 \\ H_2 \\ H_\alpha \\ H_\beta \end{bmatrix}^T \bar{\lambda} = \begin{bmatrix} H_0 \\ H_2 \\ H_\alpha \\ H_\beta \end{bmatrix}^T \begin{bmatrix} \bar{\lambda}_0 \\ \bar{\lambda}_2 \\ \bar{\lambda}_\alpha \\ \bar{\lambda}_\beta \end{bmatrix} \tag{73}$$

Combining terms, we can then write

$$\begin{aligned} \lambda^T \bar{P} \lambda &= \bar{e}_{\bar{\xi}}^T \left(H_{\bar{\xi}} \bar{P} H_{\bar{\xi}}^T\right)^{-1} \bar{e}_{\bar{\xi}} \\ &= \left(\xi - \hat{\xi}\right)^T H_{\bar{\xi}}^T \left(H_{\bar{\xi}} \bar{P} H_{\bar{\xi}}^T\right)^{-1} H_{\bar{\xi}} \left(\xi - \hat{\xi}\right) \end{aligned} \tag{74}$$

where $\bar{e}_{\xi} = H_{\xi}e$, and

$$\bar{e}_{\xi} = \begin{bmatrix} \bar{e}_0 \\ \bar{e}_{\alpha} \\ \bar{e}_{\beta} \end{bmatrix} \qquad H_{\xi} = \begin{bmatrix} H_0 \\ H_{\alpha} \\ H_{\beta} \end{bmatrix}.$$

The optimal accumulation function, then, can be written as

$$\Upsilon(\xi_t) = g(u, z_1, \dot{z}_2) - \frac{1}{2\theta} e_t^T S_t e_t \tag{75}$$

where $S(t)$ is defined as

$$S_t = H_{\xi}^T \left(H_{\xi} \bar{P} H_{\xi}^T \right)^{-1} H_{\xi}$$

and

$$\begin{aligned} \dot{g}(u, z_1, \dot{z}_2) &= \frac{1}{2}[\hat{\xi}^T \bar{Q} \hat{\xi}] - \frac{1}{2\theta}(z_1 - H_1\hat{\xi})^T V^{-1}(z_1 - H_1\hat{\xi}) \\ &\quad - \frac{1}{2\theta}[\dot{z}_2 - H_2(A_{\xi}\hat{\xi} + B_{\xi}u)]^T \bar{R}_2^{-1}[\dot{z}_2 - H_2(A_{\xi}\hat{\xi} + B_{\xi}u)] \\ g(\cdot, \cdot, \cdot)_{t=0} &= \left\| z_2 - H_2\hat{\xi}_0 \right\|^2_{(H_2 P H_2^T)^{-1}} \end{aligned} \tag{76}$$

4.2.3 The Connection Condition

Having determined expressions for the optimal return and accumulation functions, we must then determine the connection conditions which join the two solutions. As in [1], we first partition $S(t)$ as

$$S(t) = \begin{bmatrix} S_x & S_{x\alpha} & S_{x\beta} \\ S_{x\alpha}^T & S_{\alpha} & S_{\alpha\beta} \\ S_{x\beta}^T & S_{\alpha\beta}^T & S_{\beta} \end{bmatrix}$$

Then, as in the case of the reduced order robust compensator, we adjoin the constraint $z_2 = H_2\xi$. Performing the optimization with respect to the state yields:

$$\left(\frac{\partial \Psi}{\partial x_t}\right)^T + \left(\frac{\partial \Upsilon}{\partial x_t}\right)^T - \frac{1}{\theta} H_2^T \mu = 0 \tag{77}$$

$$= \Pi_t(\alpha, \beta) x_t - \frac{1}{\theta}\left[S_x(x_t - \hat{x}_t) + S_{x\alpha}(\alpha - \hat{\alpha}) + S_{x\beta}\left(\beta - \hat{\beta}\right)\right] - \frac{1}{\theta} H_2^T \mu$$

This relationship, combined with the constraint of the perfect measurements, yields the state connection condition.

Next, solving for the state coefficient parameter connection condition yields

$$(x_t - \hat{x}_t)^T S_{x\alpha} + (\alpha - \hat{\alpha})^T S_\alpha + \left(\beta - \hat{\beta}\right)^T S_{\alpha\beta}^T = \frac{\theta}{2} x_t^T \frac{\partial \Pi(\alpha, \beta)}{\partial \alpha} x_t \tag{78}$$

The control coefficient parameter connection condition is given by

$$(x_t - \hat{x}_t)^T S_{x\beta} + (\alpha - \hat{\alpha})^T S_{\alpha\beta} + \left(\beta - \hat{\beta}\right)^T S_\beta = \frac{\theta}{2} x_t^T \frac{\partial \Pi(\alpha, \beta)}{\partial \beta} x_t \tag{79}$$

The connection conditions can then be used to provide the link between the estimation problem and the control problem.

5 Conclusions

The results presented in this paper extend the existing results in robust [2, 4] and adaptive [1] control based on disturbance continuation to a class of problems where a combination of perfect and imperfect state information is available. In the case of the robust controller, this allows for the reduction of the required order of the compensator in such systems. This may be of practical importance in applications where some measurements are known nearly exactly, and compensator size is of concern.

In the adaptive case, the approach used in this paper allows for the estimation of parameters multiplying perfectly measured states, which may help to increase performance in some uncertain systems with some measurements which are known exactly.

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Appendix B
PARAMETER ROBUST GAME THEORETIC SYNTHESIS FOR THE F-18 HARV

Parameter Robust Game Theoretic Synthesis for the F-18 HARV

Charles H. Dillon* and Jason L. Speyer†
University of California, Los Angeles

Abstract

A parameter-robust game theoretic compensator has been designed for the F-18 High Angle-of-attack Research Vehicle (HARV). Modelling uncertainties due to parameter variations in the aircraft dynamics over varying flight conditions are included in the design process by using a internal feedback loop (IFL) decomposition and incorporating the resulting state space into a disturbance attenuation problem. The compensator is designed to track pilot inputs in angle-of-attack, sideslip, and stability-axis roll rate through stick and rudder pedal commands. Zero steady state error in the presence of step inputs is achieved by using a system of error coordinates which also allows for thrust vector commands to fade to zero in steady state. Linear and nonlinear simulation results are presented.

1. Introduction

The effort to date in the development of a parameter-robust controller for the F-18 high angle-of-attack research vehicle (HARV) using game theoretic controllers [1] has concentrated mainly on compensator design and evaluation. The design objective was to create a compensator capable of tracking state command inputs in the presence of uncertain dynamics at high angle-of-attack. Additionally, physical limitations of the thrust vectoring vanes dictated that the design be such that thrust vectoring be used mainly for enhancement of transient response, with aerodynamic controls being used in steady state. The approach taken in designing the control system was a linear design technique based on game disturbance attenuation [1] with parameter uncertainty included in the design process as a fictitious disturbance via an internal feedback loop (IFL) decomposition [2]. As with other linear controllers, this technique utilizes a linearized model of the aircraft about a particular flight condition. The game theoretic synthesis essentially produces a controller which reduces the sensitivity of the closed loop system to disturbances. By including parameter uncertainty as a fictitious disturbance, the sensitivity of the closed loop system to parameter variations is reduced, and this decomposition helps to extend the region in which the resulting linear control gains can be used. Therefore, by extending the usable region about each design point, fewer design points are required, thus reducing the number of gains to be stored in the final implementation of the controller.

The main analysis tool used in generating linearized models, obtaining estimates of the parameter uncertainties, and evaluating controller performance has been the Dryden F-18 HARV nonlinear batch simulation. This simulation provides a detailed nonlinear model of the aircraft, including an aerodynamic database containing data for "clean" (angle-of-attack less than 40 degrees), high alpha (angle-of-attack between 40 and 90 degrees), power approach and takeoff flight regimes. The simulation is currently hosted on a SUN workstation at UCLA. Capability for using interchangeable control laws, through a simulation control law interface provided by NASA Dryden exists within the simulation and has been used in testing a FORTRAN implementation of the linear controller with the full nonlinear aircraft dynamics.

2. Modelling Assumptions

The full nonlinear equations of motion for the F-18 HARV are given below [3,4]. The moments, denoted L, M, and N, and the forces, denoted X_w, Y_w, and Z_w, where the subscript w indicates a wind axis reference frame, have components due to gravitation, aerodynamics, and engine thrust. These forces and moments are strongly dependent on the values of state and control deflections. The vehicle mass properties are reflected in the inertia tensor, I, and vehicle mass, m. Aircraft velocity is given by V and body angular rates by p, q, and r. The matrix T_{IB} is a coordinate transformation from body to an inertial reference frame, which is a function of the Euler angles ϕ, θ, and ψ. The matrix T_{BW} is a coordinate transformation from wind to body axes, which is a function of angle-of-attack and sideslip. The rates p_S and r_S indicate stability axis

*Graduate Student, Mechanical, Aerospace and Nuclear Engineeing Dept.

† Professor, Mechanical, Aerospace, and Nuclear Engineering Dept.
Member AIAA

roll and yaw rates, respectively. The dynamics are given as:

$$\frac{d}{dt}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = I^{-1}\left\{\begin{bmatrix} L \\ M \\ N \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \cdot \begin{bmatrix} p \\ q \\ r \end{bmatrix}\right\}$$

$$\frac{d}{dt}\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \dfrac{\sin\phi}{\cos\theta} & \dfrac{\cos\phi}{\cos\theta} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\dot{\alpha} = q - p_s \tan\beta + \frac{Z_w}{mV\cos\beta}$$

$$\dot{\beta} = \frac{Y_w}{mV} - r_s \qquad (1)$$

$$\dot{V} = \frac{X_w}{m}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ -h \end{bmatrix} = T_{IB}\, T_{BW} \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p_s \\ r_s \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix}$$

For the controller design, a linearized model was extracted from the nonlinear vehicle dynamics at a particular flight condition. A 5-state system was chosen to represent the dynamics of interest in controller design which included angle-of-attack α,, sideslip β, and roll, pitch and yaw body angular rates (p, q, r). Control surfaces, a total of 8, included leading and trailing edge flaps (δ_{LEF}, δ_{TEF}), rudder (δ_{RDR}), differential and collective horizontal tail (δ_{HT}), differential aileron (δ_{AIL}), and pitch and yaw thrust vectoring. The linearized model then had the form as shown below. This model was generated using the linearization routine in the nonlinear batch simulation at a given trim condition. The condition selected for controller design was at an angle-of-attack of 50 degrees and an altitude of 25000 ft. The states of the linearized model, then, represent perturbations of the nonlinear model states about their nominal trim values. Actuator dynamics were not included in the state space used for design. The linear dynamical system is then represented by:

$$\dot{x} = Ax + Bu$$
$$x = [\alpha, \beta, p, q, r]^T \qquad (2)$$
$$u = \left[\delta_{HT_p}, \delta_{LEF_p}, \delta_{TEF_p}, \delta_{TVC_p}, \delta_{AIL_r}, \delta_{HT_r}, \delta_{RDR_y}, \delta_{TVC_y}\right]^T$$

where the A and B matrices contain the linearized force and moment coefficients at the given flight condition.

A vital part of the controller design was in determination of the model uncertainty for use in the synthesis procedure. To accomplish this, several trim conditions about the nominal were selected and linear models were generated at these points. Linear model coefficients were then compared at the varying flight conditions to determine which coefficients represented the greatest sources of uncertainty. Upon examination of these variations, it was found that many of the coefficients varied in magnitude over the ranges considered, but some varied in sign as well. The state coefficients that seemed to exhibit the greatest amount of uncertainty were L_β, M_α, and N_β. The control coefficients which varied the most significantly were $Y_{\delta AIL_r}$, $Y_{\delta HT_r}$, $L_{\delta RDR_y}$ and $N_{\delta HT_r}$. A sample of the variations in these coefficients is shown in Figs. 1-6.

3. Compensator Design and Implementation

3.1 Design of State Space for Compensator Design

The goal in designing the compensator for the linearized dynamic system was to track state command inputs such that steady state errors due to step inputs would go to zero. This was accomplished by using a system of error coordinates, $e_1 = x - x_c$, where x_c is the input vector step command. By differentiating the error coordinates, the resulting state space is a linear system having the derivative of the control surface deflections as a controlling input, which decays to zero in steady state in the presence of step commands [5,6]. Also, it was desired that the thrust vector control commands be driven to zero in steady state to avoid damaging the thrust vectoring control vanes. This was accomplished by defining as a state a subset of the controls, $v_1 = B_w u$, where Bw is defined such that $v_1 = [\delta_{TVCp}, \delta_{TVCy}]^T$. The resulting linear state equations were then given by:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{v}_1 \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ v_1 \end{bmatrix} + \begin{bmatrix} 0 \\ B \\ B_w \end{bmatrix} \dot{u} \qquad (3)$$

To model parameter variations in the linearized state space, an internal feedback loop (IFL) decomposition was used [2,7]. This decomposition is represented by the system :

$$\begin{gathered} x_e \equiv \left[e_1^T e_2^T v_1^T \right]^T \\ \dot{x}_e = (A_0 + \Delta A)\, x_e + (B_0 + \Delta B)\, \dot{u} \\ \Delta A = D\, L_a(\varepsilon)\, E \qquad \Delta B = F\, L_b(\varepsilon)\, G \end{gathered} \quad (4)$$

The matrices $L_a(\varepsilon)$ and $L_b(\varepsilon)$ contain functions which describe the form of the parameter variations, with ε in this case representing uncertainty in the linearized state and control coefficients.. By decomposing the parameter variations as such, the system can be rewritten in a form where the parameter variations act as fictitious disturbances to the nominal system. This resulting system is then written as:

$$\begin{gathered} \dot{x}_e = A_0 x_e + B_0 \dot{u} + \Gamma w \\ y_1 = \begin{bmatrix} E \\ 0 \end{bmatrix} x_e + \begin{bmatrix} 0 \\ G \end{bmatrix} \dot{u} \\ w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} L_a(\varepsilon) & 0 \\ 0 & L_b(\varepsilon) \end{bmatrix} y_1 \\ \Gamma = [D \;\; F] \end{gathered} \quad (5)$$

where Γ is a new disturbance input matrix which includes the ficticious disturbances representing parameter uncertainties, y_1 is an output vector associated with the parameter uncertainty and w is considered as the disturbance vector.

3.2 The Disturbance Attenuation Problem

Once the system has been written in the above form, a controller can be designed which reduces the sensitivity of the closed loop system to parameter variations, acting as disturbances to the nominal linearized system. This is done by first constructing a disturbance attenuation function, which is essentially the ratio of desired performance outputs to disturbance inputs. For all admissible disturbances, a controller must then be found which bounds this function:

$$\sup_{w \in L_2} \left[\int_0^{t_f} \left(\rho y^T y + y_1^T y_1 \right) dt \Big/ \int_0^{t_f} w^T w \, dt \right] < \gamma^2 \quad (6)$$

The vector y represents a linear combination of states and controls to be weighted quadratically to attain a desired level of performance and ρ is a scalar which is used to adjust the relative weighting between performance and sensitivity. The performance output y is represented by:

$$y = \begin{bmatrix} C \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ C_1 \end{bmatrix} \dot{u} \quad (7)$$

The disturbance attenuation problem can then be converted to a differential game problem with a quadratic performance index given by:

$$J(u,w,t_f) = \int_0^{t_f} \left(y^T y + y_1^T y_1 - \gamma^2 w^T w \right) dt \quad (8)$$

The solution of this problem, as presented in [1] is obtained by allowing final time to extend to infinity, and then solving the resulting algebraic Riccati equation (ARE):

$$\begin{gathered} A_0^T \overline{\Pi} + \overline{\Pi} A_0 - \overline{\Pi} \left(B_0 R^{-1} B_0^T - \frac{1}{\gamma^2} \Gamma \Gamma^T \right) \overline{\Pi} + Q = 0 \\ Q = \rho C^T C + E^T E \\ R = \rho C_1^T C_1 + G^T G \end{gathered} \quad (9)$$

This algebraic Riccati equation, however, has a sign indefinite quadratic term. The parameter γ must be adjusted until a solution exists. As gamma extends to infinity, this becomes equivalent to the standard linear quadratic regulator problem. The resulting controller obtained by solving the ARE, then, is a linear combination of the states of the form:

$$\dot{u} = -R^{-1} B_0^T \overline{\Pi}\, x \quad (10)$$

For the error coordinate system, this results in the derivative of the control being a function of the state errors, their derivatives, and the thrust vector commands. The control, then, results in a linear combination of the error, the integral of the error, and the integral of the thrust vector commands.

$$u = -\left[K_{e_1} \;\; K_{e_2} \;\; K_{v_1} \right] \left[\int e_1^T \;\; e_1^T \;\; \int v_1^T \right]^T \quad (11)$$

The resulting compensator is essentially a proportional plus integral system. A block diagram of the implementation of the above controller is shown in Figure 7. State command inputs are obtained as a function of pilot longitudinal and lateral stick and rudder pedal inputs. The baseline implementation commands angle-of-attack as a function of longitudinal stick, stability axis roll rate as a function of longitudinal stick, and sideslip as a function of rudder pedal. This combination was selected to be somewhat consistent with the baseline Research Flight Control System [8]. It should be noted, however, that any combination of the 5 states may be commanded by combinations of these pilot inputs if desired. For example, a pitch rate command system would be just as easily implemented, as the baseline implementation simply uses zero as the commanded pitch rate. A first order prefilter was added with natural frequency of 5 rad/s to effectively limit the bandwidth of command inputs so as to avoid actuator rate saturation.

The selection of the state and control weightings C and C_1 was made such that reasonably fast step response to command inputs could be achieved without causing actuator saturation. A baseline controller was first designed without including uncertainty in order to determine a reasonable set of state and control weightings to be used. Once these were determined, the IFL decomposition was used to include uncertainty effects in the controller design. The parameters ρ and γ were adjusted accordingly to produce a controller which bounds the associated disturbance attenuation function and provides a reasonably good balance between system performance and parameter sensitivity.

4. Simulation Results

In order to evaluate the controller design, simulations were performed using both the linearized model and the full nonlinear batch simulation model. To generate nonlinear simulation results, the controller was implemented as a FORTRAN subroutine and connected to the batch simulation via the control law interface provided by NASA. Linear simulation results were generated using linearized models in MATLAB. A nominal trim flight condition at angle-of-attack of 50 degrees and altitude of 25000 feet was selected to perform the evaluations. Step inputs in angle of attack and doublets in stability axis roll rate were performed in both the linear and nonlinear simulations. Comparisons between control designs synthesized with and without the presence of parameter uncertainty were made in both the linear and nonlinear case. Gains generated with the effects of parameter uncertainty used a value of 1000 for the parameter γ. Results from the nonlinear simulation were then compared to linear results to verify the effectiveness of the controller at flight conditions varying from the nominal trim condition.

4.1 Linear Simulation Results

Linear simulation results were generated to determine the response of the controller in the linear region about the nominal trim operating point. State and control performance weightings were adjusted in the design process to provide adequate time response to step inputs in the commanded states, without requiring overly large commands from any of the controls. As the effect of parameter uncertainty was added to the design, comparisons could be made to show the resulting effect on time response.

Figures 8-9 show time responses of the linear system using gains generated with and without parameter uncertainty. Linear step responses in angle-of-attack and doublet responses in stability axis roll rate are presented. Linear responses to longitudinal step inputs are virtually identical with and without uncertainty included in the design, while there is a slight difference in lateral response. This is mainly due to the fact that most of the uncertainty accounted for in the design process was in the lateral coefficients.

4.2 Nonlinear Simulation Results

Responses were generated in the nonlinear batch simulation starting from the nominal trim condition of 50 degrees angle of attack at 25000 feet altitude. Step responses in angle-of-attack were fairly consistent near the nominal point, but had some difficulty achieving the higher angle-of-attack flight conditions due to actuator saturations. This is illustrated in Figure 10. Lateral responses, shown in Figures 11-12, showed only small differences between the design with and without the effects of uncertainty. The main difference between the two can be seen by examining the actuator responses shown in Figures 13-16. The parameter robust controller tends to use the thrust vectoring, whose effect is known with greater certainty, than the aerodynamic controls, which are less well-known.

5. Conclusions

A parameter robust controller based on game theoretic synthesis has been designed for the F-18 HARV and implemented in nonlinear batch simulation. The goal of this design was to be able to track pilot command inputs in the presence of parameter uncertainties. A linear controller was designed and game theoretic synthesis used to extend the valid range of the linear controller. The use of error coordinates in the controller design provided for a compensator which achieves zero tracking error to step command inputs in steady state. Also, the compensator design allows for any combination of angle-of-attack, sideslip, roll, pitch and yaw body rates to be commanded with zero steady state error.

From simulation results, it was seen that this controller produced some improvement in the lateral response, although limitations due to actuator effectiveness and saturations at the high angle-of-attack flight conditions considered limited the amount of improvement which could be observed. The most significant effect of the use of the parameter robust controller is seen mainly in the use of different actuators to obtain similar responses. That is, actuators which influence the behavior of the dynamic response with more certainty, such as thrust vectoring, are used more heavily than aerodynamic controls which have less well-known effects at high angle-of-attack.

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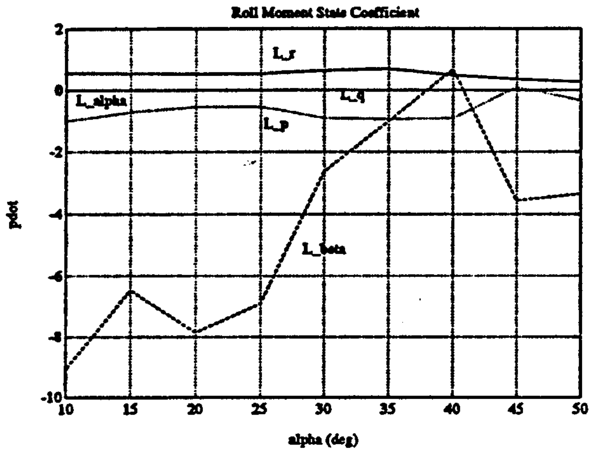

Figure 1. Variation in Roll Moment State Coefficients with Angle of Attack

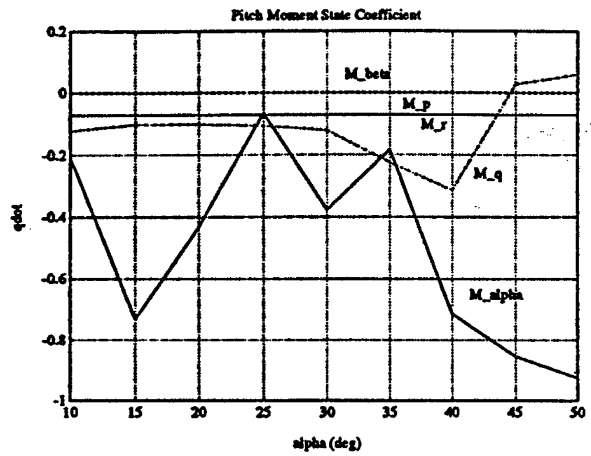

Figure 2. Variation in Pitch Moment State Coefficients with Angle of Attack

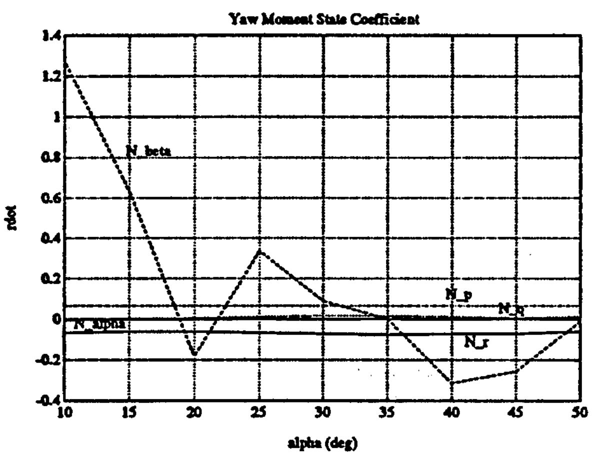

Figure 3. Variation in Yaw Moment State Coefficients with Angle of Attack

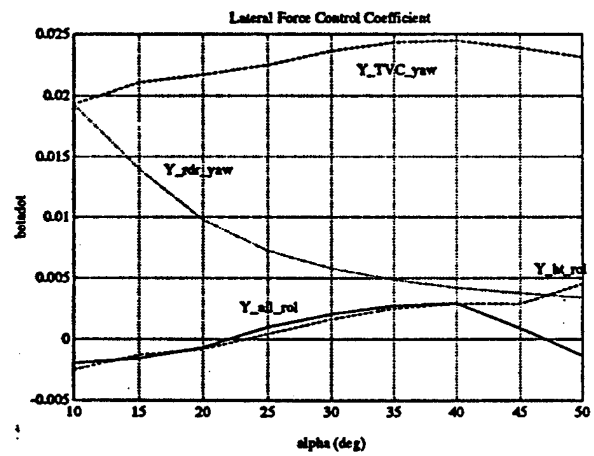

Figure 4. Variation in Lateral Force Control Coefficients with Angle of Attack

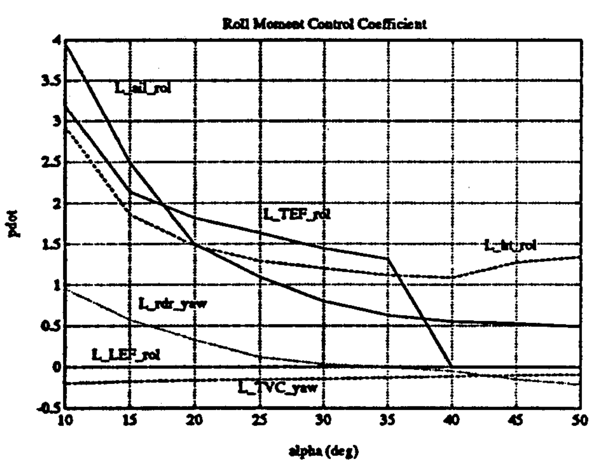

Figure 5. Variation in Roll Moment Control Coefficients with Angle of Attack

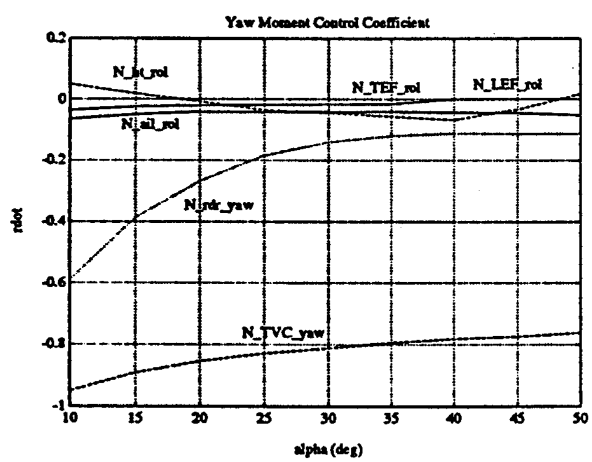

Figure 6. Variation in Yaw Moment Control Coefficients with Angle of Attack

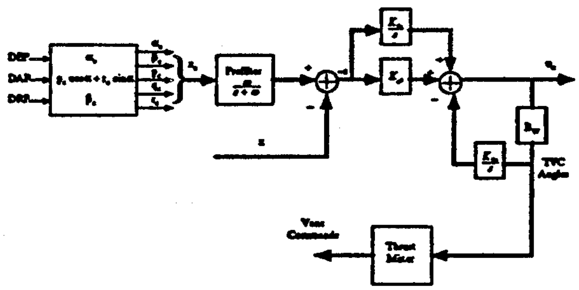

Figure 7. Controller Implementation

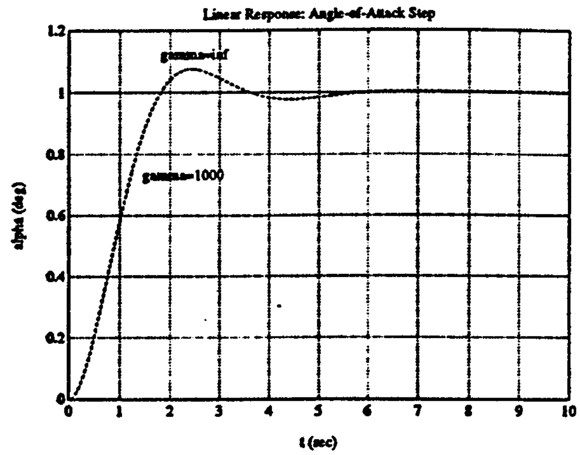

Figure 8. Angle-of-Attack Linear Step Response

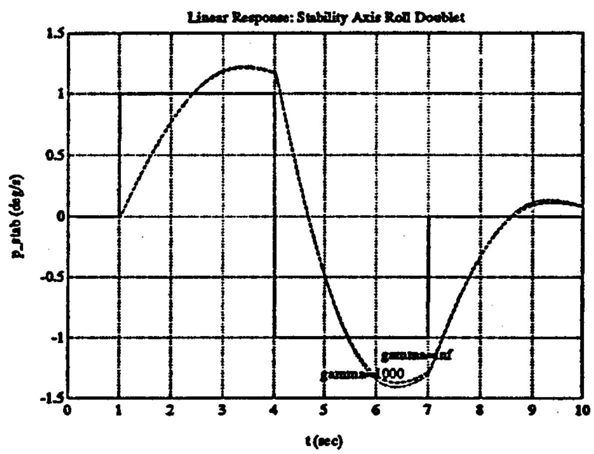

Figure 9. Lateral Doublet Maneuver Linear Response

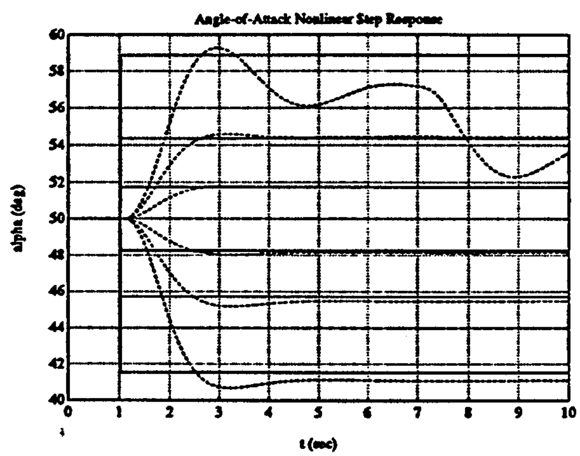

Figure 10. Angle of Attack Nonlinear Step Response

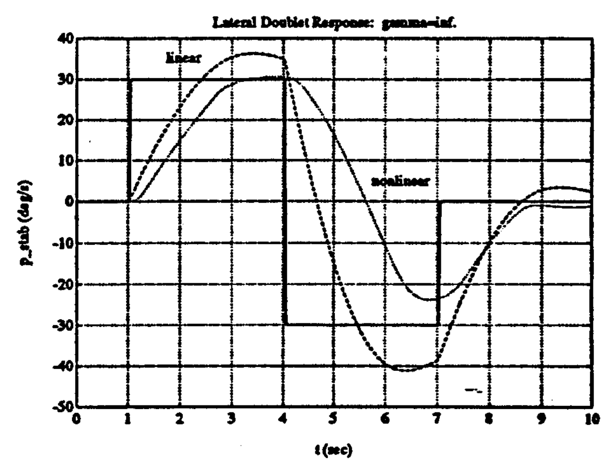

Figure 11. Nonlinear Lateral Doublet Response : PRGTS Design Without Uncertainty

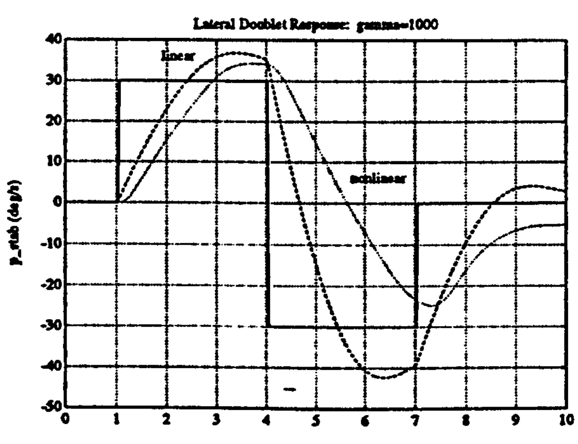

Figure 12. Nonlinear Lateral Doublet Response : PRGTS Design With Uncertainty

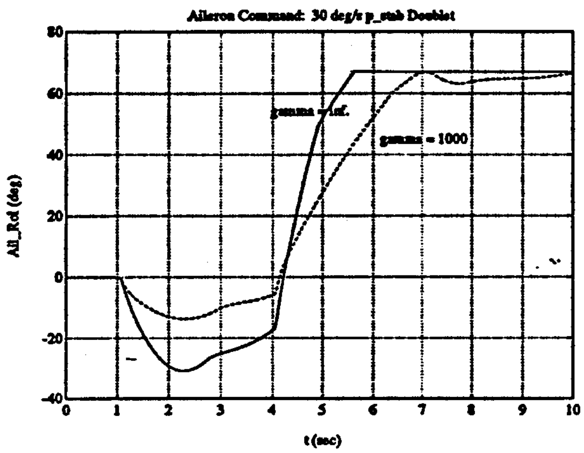

Figure 13. Aileron Command: 30 deg/s Stbility Axis Roll Rate Doublet

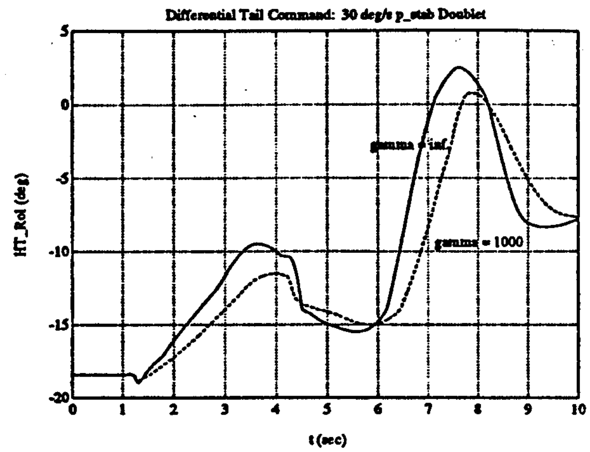

Figure 14. Differential Tail Command: 30 deg/s Stbility Axis Roll Rate Doublet

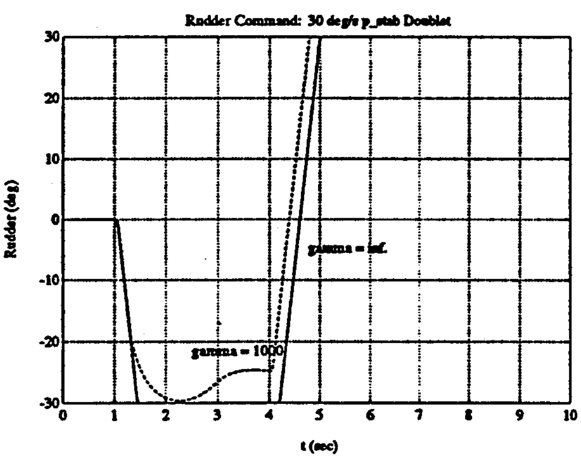

Figure 15. Rudder Command: 30 deg/s Stbility Axis Roll Rate Doublet

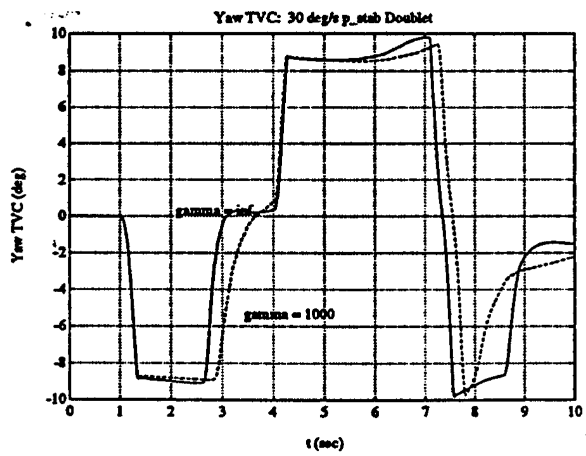

Figure 16. Yaw Thrust Vector Command: 30 deg/s Stbility Axis Roll Rate Doublet

Appendix C

A DISTURBANCE ATTENUATION ADAPTIVE CONTROLLER FOR THE LONGITUDINAL MODE OF THE F-18 HARV

Disturbance Attenuation Approach to Adaptive Control Applied to Longitudinal Flight Control of the F-18 HARV

C. H. Dillon and J. L. Speyer
University of California, Los Angeles

Abstract

The disturbance attenuation approach to controlling systems subject to external disturbances and uncertain coefficients has recently been under investigation. In this approach, a ratio of performance outputs to disturbance inputs is formed, and a controller is found which maintains this ratio below a certain bound. This disturbance attenuation problem is solved by converting the disturbance attenuation function to a performance index of a dynamic game problem. This game problem is then solved using dynamic programming. The resulting compensator structure is adaptive in that estimates of uncertain coefficients are calculated on line, and robust in that the control is a function of the worst case state and parameter values. This paper demonstrates the application of such a compensator to the problem of longitudinal flight control with uncertain aerodynamic control moment coefficient and well known thrust vector control. Step responses in angle of attack are presented which illustrate the effectiveness of the robust adaptive compensator.

1 Introduction

Over the past several years, techniques have been developed for controlling uncertain linear systems subject to external disturbances by considering a disturbance attenuation problem. In this approach, a measure which is essentially the ratio of norms of performance outputs to disturbance inputs is created, and a robust compensator is sought which bounds this ratio below some limit. In [1, 2], this problem was approached by converting this disturbance attenuation function to a performance index and then using a game theoretic approach to find the minimizing control for the worst case maximizing disturbance. This approach extended the results of H_∞ analysis to include not only time invariant systems on infinite intervals, but time-varying systems on finite intervals as well.

In the work presented in [3], this disturbance attenuation approach was applied to a class of problems in which uncertainty exists in one or more parameters in the system control coefficient matrix. Using a dynamic programming approach suggested in [1], the problem was split into two parts. Optimizing from current to final time yielded a controller as a function of the states and parameters and an optimal return function which represents the cost to go from the current time to the final time. Optimizing from initial to current time yielded an estimator structure which provides estimates of the current state and parameter values based on past measurements up to the current time as well as an optimal accumulation function which represents the cost accumulated up to the current time. An algebraic "connection condition" was then determined by maximizing the sum of the optimal return function and the optimal accumulation function with respect to the states and parameters at the current time. This then resulted in a compensator structure which is both robust in that it chooses a control based on the worst case disturbances and parameter uncertainties and adaptive in that the uncertain parameters are estimated using available measurements.

The problem of flight control design at high angles of attack presents a natural application of such robust and adaptive compensators. In [4], a robust controller was designed for high angle of attack flight conditions of the F-18 HARV (High Angle-of-Attack Research Vehicle) aircraft. The compensator was designed for zero steady state tracking of pilot inputs by augmenting the state space with integral error states. Additionally, due to the physical limitations of the thrust vectoring hardware on the aircraft, additional "washout" states were added so that thrust vector commands were faded to zero in steady state. The robust compensator design was then used to effectively expand the usable region of the linear controller about each design point.

One difficulty which arises with such a design is that parameters in the linearized system may change rapidly and in some cases switch signs over dynamically varying flight conditions. By estimating the parameters which tend to vary the most and/or have the greatest effect on system performance, it may be possible to increase the overall performance of the compensator as parameter values become more well known. The unique advantage of a robust adaptive compensator such as that which is presented in this paper is that by forming the control based on the worst case values of state and parameters, the compensator can effectively use the controls whose coefficients are better known until enough measurements have been taken to reduce the uncertainty in the unknown coefficients to the point where the associated control can be used with confidence.

In the example presented in this paper, the longitudinal dynamics of the F-18 HARV are considered at a given flight condition. The plant states are angle of attack, α, and pitch rate, q. The controls to be used are elevator deflection, δ_e, and thrust vectoring, δ_{TVC}. The problem is to follow step commands in angle of attack with zero steady state tracking error while fading the thrust vectoring commands to zero in steady state due to the hardware restrictions of the paddles used for thrust vectoring on the F-18 HARV. To demonstrate the behavior of the robust adaptive compensator, the moment coefficient due to elevator deflection, M_{δ_e}, is considered to be unknown and is to be estimated on line. Results are shown which illustrate the effect of the disturbance attenuation bound.

2 Disturbance Attenuation Problem

The problem of disturbance attenuation is one of finding a control which limits the effects of all admissable disturbances and uncertainties on the compensated system. A disturbance attenuation function is formed which is essentially a ratio of the norms of performance outputs over disturbance inputs. The problem, then, is to find a positive parameter θ such that this disturbance attenuation function is bounded. This function can be written as:

$$D = \frac{\|\bar{y}\|^2}{\|\bar{w}\|^2} \leq \frac{1}{\theta} \quad \theta > 0 \tag{1}$$

where the measures of performance outputs, $\|\bar{y}\|^2$ and $\|\bar{w}\|^2$ are defined as

$$\|\bar{y}\|^2 = \|x(t_f)\|^2_{Q_f} + \int_0^{t_f} \left(\|x\|^2_Q + \|u\|^2_R \right) d\tau \tag{2}$$

$$\|\bar{w}\|^2 = \|\xi(0)\|^2_{P_0^{-1}} + \int_0^{t_f} \left(\|w\|^2_{W^{-1}} + \|v\|^2_{V^{-1}} \right) d\tau \tag{3}$$

where x represents the states, u the controls, w the plant input disturbance, v the state measurement noise, and ξ represents the augmented state defined as $\xi = \left[\begin{array}{cc} x^T & \beta^T \end{array} \right]^T$, where β represents the unknown control coefficient matrix parameters. The dynamic system under consideration is of the form:

$$\dot{x} = Ax + B(\beta)u + \Gamma w \tag{4}$$

$$z = Hx + v \tag{5}$$

$$\tag{6}$$

where

$$B(\beta) = B_0 + \sum_{j=1}^{k} B_j \beta_j \tag{7}$$

$$\dot{\beta} = 0 \tag{8}$$

To approach this problem, as in [2, 3], we reformulate the disturbance attenuation problem as a differential game problem, with performance index given by:

$$J = \frac{1}{2} \left\{ \|\bar{y}\|^2 - \frac{1}{\theta} \|\bar{w}\|^2 \right\} \leq 0 \tag{9}$$

For a given value of θ, then, the problem becomes one of finding the control u which minimizes this cost in the presence of the worst case maximizing disturbance inputs provided by initial conditions, $\xi(0)$, and state and measurement noise (w, v).

3 Dynamic Programming Solution

It is shown in that the minimax problem associated with the performance index given by (9) reduces to a saddle point problem so that the operations of minimization and maximization can be interchanged [3, 5]. This leads to a dynamic programming solution with an accompanying decomposition, which is further developed in [3]. The first part of this decomposition, a control subproblem, defines an optimal return function, $\Psi(x_t, \beta)$, which represents the cost to go from current time, t, to final time, t_f. The second part, a filtering subproblem, defines an optimal accumulation function, $\Upsilon(x_t, \beta)$, which represents the cost accumulated from initial time, 0, to current time, t. The dynamic programming problem, then, is to find the values of state and parameters, $\breve{x}, \breve{\beta}$ such that:

$$\Upsilon\left(\breve{x}_t, \breve{\beta}\right) + \Psi\left(\breve{x}_t, \breve{\beta}\right) \geq \Upsilon(x_t, \beta) + \Psi(x_t, \beta) \quad \forall x_t, \beta \tag{10}$$

3.1 The Control Subproblem

The control subproblem represents the task of finding the control, u, which minimizes the cost to go from the current time, t, to the final time, t_f. To accomplish this, we first write this cost as:

$$J_c[t, t_f] = \frac{1}{2}\left\{\|x(t_f)\|^2_{Q_f} + \int_t^{t_f}\left[\|x\|^2_Q + \|u\|^2_R - \frac{1}{\theta}\left(\|w\|^2_{W^{-1}} + \|v\|^2_{V^{-1}}\right)\right] d\tau\right\} \tag{11}$$

Appending the dynamics (4) to the cost and solving as in [3] yields an expression for the minimizing control, u, and maximizing disturbance, w, as a function of the states and parameters. The parameter dependence arises through the associated Riccati equation, which is calculated using the control coefficient matrix, $B(\beta)$ evaluated for the given values of the parameters, β. Thus, we have:

$$\begin{aligned} \breve{u}(\tau) &= -R^{-1}B(\beta)^T\Pi(\beta, \tau)x(\tau) && (12)\\ \breve{w}(\tau) &= \theta W\Gamma^T\Pi(\beta, \tau)x(\tau) && (13)\end{aligned}$$

where the matrix $\Pi(\beta, \tau)$ is defined as the solution of the Riccati equation

$$\begin{aligned} -\dot{\Pi} &= A^T\Pi + \Pi A + Q - \Pi\left(B(\beta)R^{-1}B(\beta)^T - \theta\Gamma W\Gamma^T\right)\Pi && (14)\\ \Pi(t_f) &= Q_f \end{aligned}$$

Substituting the expressions obtained for the control and disturbance back into the performance index (11) gives the desired expression for the optimal return function:

$$\Psi(x_t, \beta) = \frac{1}{2} x_t^T \Pi(\beta) x_t \tag{15}$$

The control which arises from this portion of the solution, then, is based upon full information of the state and parameter values. This solution has been further extended to allow infinite final time in [5].

3.2 The Filtering Subproblem

The second part of the problem considered is that of solving the optimization problem in reverse time from time 0 to the present time, t. To do this, the cost is first written in terms of the augmented state, ξ.

$$J_f[0,t] = \frac{1}{2}\left\{-\frac{1}{\theta}\left\|\xi(0) - \hat{\xi}_0\right\|^2_{P_0^{-1}} + \int_0^t \left[\|\xi\|^2_{\bar{Q}} + \|u\|^2_R - \frac{1}{\theta}\left(\|w\|^2_{W^{-1}} + \left\|z - \bar{H}\xi\right\|^2_{V^{-1}}\right)\right] d\tau\right\} \tag{16}$$

where

$$\bar{Q} = \begin{bmatrix} Q & 0_{n\times p} \\ 0_{p\times n} & 0_{p\times p} \end{bmatrix}$$

and the dynamics of the system (4) are rewritten in terms of the augmented state, ξ as:

$$\dot{\xi} = \bar{A}\xi + \bar{B}u + \bar{\Gamma}w \tag{17}$$

$$z = \bar{H}\xi + v \tag{18}$$

with $\bar{A}$, $\bar{B}$, $\bar{\Gamma}$, and $\bar{H}$ defined as

$$\bar{A} = \begin{bmatrix} A & B_1 u & \cdots & B_k u \\ 0 & & 0 & \end{bmatrix} \quad \bar{B} = \begin{bmatrix} B_0 \\ 0 \end{bmatrix} \quad \bar{\Gamma} = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} \quad \bar{H} = \begin{bmatrix} H & 0 \end{bmatrix}$$

Adjoining the augmented dynamics, (17) to the performance index (16) and solving the optimization problem results in an equation for an estimator which gives estimates of the state and parameter values at time t based on the information contained in the measurements, z, up to time t. This estimator is represented by:

$$\begin{aligned} \dot{\hat{\xi}} &= \left(\bar{A} + \theta P\bar{Q}\right)\hat{\xi} + \bar{B}u + P\bar{H}^T V^{-1}\left(z - \bar{H}\xi\right) \\ \hat{\xi}(0) &= \hat{\xi}_0 \end{aligned} \tag{19}$$

where the matrix P is the solution of a Riccati equation given by:

$$\begin{aligned} \dot{P} &= \bar{A}P + P\bar{A}^T + \bar{\Gamma}W\bar{\Gamma}^T - P\left(\bar{H}^T V^{-1}\bar{H} - \theta\bar{Q}\right)P \\ P(0) &= P_0 \end{aligned} \tag{20}$$

Combining these relations with the performance index as in [3] gives the desired relationship for the optimal accumulation function:

$$\Upsilon(x_t, \beta) = g\left(u_0^t, \hat{x}_0^t\right) - \frac{1}{2\theta}e^T S e \tag{21}$$

where

$$\begin{aligned} \dot{g} &= \frac{1}{2}\left(u^T R u + \hat{x}^T Q \hat{x}\right) - \frac{1}{2\theta}(z - H\hat{x})^T V^{-1}(z - H\hat{x}) \\ e &= \xi_t - \hat{\xi}_t \\ S &= P_t^{-1} \end{aligned}$$

3.3 The Connection Condition

The final piece in forming the adaptive compensator is to perform the maximization of the sum of the optimal return function (15) and the optimal accumulation function (21) at time t. That is, we wish to find the worst case state and parameter values, x_t^* and β^* which are found from the maximization

$$\max_{x_t, \beta}\left[\Psi(x_t, \beta) + \Upsilon(x_t, \beta)\right] \tag{22}$$

This maximization can be simplified somewhat by first solving the maximization with respect to the state x_t which gives an algebraic relation as a function of the parameter β. Partitioning the matrix S as

$$S = \begin{bmatrix} S_{xx} & S_{x\beta} \\ S_{\beta x} & S_{\beta\beta} \end{bmatrix}$$

the expression for the worst case state x_t^* as a function of the parameter β is then given by

$$x_t^* = \left[\theta\Pi(\beta) - S_{xx}\right]^{-1}\left[S_{x\beta}\left(\beta - \hat{\beta}\right) - S_{xx}\hat{x}_t\right] \tag{23}$$

with the requirement that

$$\theta\Pi(\beta) - S_{xx} < 0 \quad \forall\beta \tag{24}$$

The maximization problem can then be solved as a function of the parameter only. It should be noted, however, that since the Riccati solution $\Pi(\beta)$ is dependent on β, the resulting function to be maximized becomes quite nonlinear and may have more than one local maximum. It is possible that at some time t there could be two peaks of equal magnitude. Should such a case arise, it should be noted that although the worst case state and parameter values at these peaks will be different, the resulting control will be the same [3].

4 Application to Longitudinal Flight Control

The basic dynamic system to be considered in this example is a system consisting of two states (angle of attack, α, and pitch rate, q) and two controls (elevator deflection, δ_e, and thrust vector command, δ_{TVC}). The basic dynamics for this system are obtained by linearizing the dynamics of the airplane about a particular trim condition. For this particullar example, a flight condition at an altitude of 25,000 feet and angle of attack of 10 degrees was selected. The basic dynamic system, then, can be written in the form:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha & Z_q \\ M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} Z_{\delta_e} & Z_{\delta_{TVC}} \\ M_{\delta_e} & M_{\delta_{TVC}} \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_{TVC} \end{bmatrix} + \begin{bmatrix} w_\alpha \\ w_q \end{bmatrix} \tag{25}$$

$$z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} v_\alpha \\ v_q \end{bmatrix} \tag{26}$$

In formulating the dynamic system to be controlled, we first consider the objectives of the control design. Primarily, we would like to be able to track step commands in angle of attack, α, with zero steady state error. As a secondary objective, we would like to fade the thrust vector control command, δ_{TVC}, to zero in steady state to avoid damaging the paddles which are used as actuators for thrust vectoring on the F-18 HARV. Also, since the control effectiveness of the elevator can change at varying flight conditions, we consider M_{δ_e} as the uncertain parameter to be estimated on line.

To accomplish our first objective, we formulate a change of variables and define the error coordinate e_α as the error between actual angle of attack, α, and commanded angle of attack, α_c.

$$e_\alpha = \alpha - \alpha_c \tag{27}$$

To track step commands in α with zero steady state error, a constant value of the control deflection δ_e will be required in steady state. To assure that the problem remains well posed we must form a new state space by differentiating the error so that the control in the dynamic system used in the design synthesis is actually the derivative of the actual physical control. This assures that we have a state space for which the performance index will remain finite as final time becomes infinite [6].

Next, we need to incorporate a means of fading the thrust vector command, δ_{TVC}, to zero in steady state. To do this, we include δ_{TVC} in the state space which we can then weight in our performance index so that it is driven to zero in steady state. However, we must also note that the system must remain controllable for all values of the parameter β. To assure that this condition is true, we redefine δ_e as two controls,

$$\delta_e = (\delta_e)_{known} + (\delta_e)_{unknown}$$

where $(\delta_e)_{known}$ is multiplied by a fixed value of the control coefficient, $(M_{\delta_e})_0$ which allows the system to remain controllable for all values of the unknown value of M_{δ_e} which multiplies $(\delta_e)_{unknown}$. Since, in reality, the value of M_{δ_e} is uncertain, we choose the control weightings in the performance index so that the cost associated with using $(\delta_e)_{known}$ is much greater than the cost associated with using $(\delta_e)_{unknown}$.

The dynamic system used in the solution of the control subproblem, then, is defined as in (4) with

$$x = \begin{bmatrix} e_\alpha \\ q \\ \dot{\alpha} \\ \dot{q} \\ \delta_{TVC} \end{bmatrix} \qquad u = \begin{bmatrix} \left(\dot{\delta}_e\right)_{unknown} \\ \left(\dot{\delta}_e\right)_{known} \\ \dot{\delta}_{TVC} \end{bmatrix}$$

and

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & Z_\alpha & Z_q & 0 \\ 0 & 0 & M_\alpha & M_q & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Z_{\delta_e} & Z_{\delta_e} & Z_{\delta_{TVC}} \\ M_{\delta_e} & (M_{\delta_e})_0 & M_{\delta_{TVC}} \\ 0 & 0 & 1 \end{bmatrix} \quad \Gamma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

For the filtering subproblem, the augmented state is simply

$$\xi = \begin{bmatrix} x \\ M_{\delta_e} \end{bmatrix}$$

As an additional note, since for the objectives we have specified, the pitch rate q does not need to be controlled, we effectively ignore that state when formulating the controller. To be able to control q we would either need to include another aerodynamic control such as flap deflection, or allow the thrust vectoring command to attain a nonzero steady state value.

In solving the maximization problem to obtain the connection condition, we note that the state δ_{TVC} is actually something which we calculate directly and its dynamics are decoupled from the rest of the states, so that we may simply adjoin the constraint $\delta_{TVC} = \hat{\delta}_{TVC}$ to (22). Partitioning the Riccati matrix Π as

$$\Pi = \begin{bmatrix} \Pi_{xx} & \Pi_{xu} \\ \Pi_{ux} & \Pi_{uu} \end{bmatrix}$$

the connection condition for the worst case state becomes

$$x_t^* = \left[\theta \Pi_{xx}(\beta) - S_{xx}\right]^{-1} \left[S_{x\beta}\left(\beta - \hat{\beta}\right) - S_{xx}\hat{x}_t - \theta \Pi_{xu}\delta_{TVC}\right] \tag{28}$$

5 Simulation Results

To demonstrate the behavior of the adaptive controller at varying values of the parameter θ, step responses in angle of attack were simulated. To emphasize the effect of θ on the behavior of the compensated system, the initial parameter estimate was taken to have the opposite sign of the true value of M_{δ_e}. A step input of 10 degress from the initial trim condition at 10 degrees angle of attack and $25,000$ feet altitude was commanded. The linearized system coefficients at this flight condition are given in Table 1. To best demonstrate the effect of the adaptive controller, the initial estimate of M_{δ_e} was taken to be 2.0.

Coefficient	Value
Z_α	-0.3367
Z_q	0.9976
M_α	-0.2065
M_q	-0.1229
Z_{δ_e}	-0.0693
$Z_{\delta_{TVC}}$	-0.0278
M_{δ_e}	-2.7320
$M_{\delta_{TVC}}$	-1.4747

Table 1: Linearized System Coefficients, $\alpha = 10$ deg, $h = 25{,}000$ ft

In Figure 1, 10 degree step responses in angle of attack are presented. In the case of $\theta = 0$, the response initially moves in the opposite direction before rising to the commanded angle of attack and eventually settling to a zero steady state error. As θ is increased, hence the disturbance attenuation bound decreased, the step responses become faster and no longer exhibit the characteristic of an initial response in the negative direction. This is also reflected in the pitch rate responses shown in Figure 2.

To understand why this increase in θ improves the step response, we can examine the control commands. First, examining the elevator deflection rate,

$\dot{\delta}_e$, shown in Figure 3 and elevator deflection, δ_e, shown in Figure 4 we see that for $\theta = 0$ a great deal of effort is applied using this control, although it is actually moving in the wrong direction due to the erroneous initial estimate of M_{δ_e}. As θ is increased, though, we see that the controller initially uses very little elevator. Conversely, in Figures 5 and 6, we see that at $\theta = 0$ that a relatively small amount of thrust vectoring is used, but as θ is increased, thrust vectoring is used more heavily initially before fading to zero in steady state. In essence, the controller seems to hedge against using the elevator initially, due to its uncertain effects, while relying more heavily upon the thrust vectoring, whose effect is known with much greater certainty.

The estimated values of M_{δ_e} shown in Figure 7 reflect the heavier utilization of elevator for $\theta = 0$. As θ is increased from 0, the estimator response is initially slower, but becomes faster as θ increases. The key factor in the behavior of the controller, however, lies in the worst case parameter value, $M^*_{\delta_e}$, plotted in Figure 8. Initially, the true value of the parameter is highly uncertain, which is reflected in the "variance" of the parameter M_{δ_e}, shown in Figure 9 and the "cross covariance" between the parameter and state, as shown in Figure 10. At some point, as the parameter value becomes more well known, this worst case value begins to follow the estimate, $\hat{M}_{\delta_e}$, and the controller then begins to use the elevator deflection more confidently.

The point in time at which the controller begins to follow the estimated value of M_{δ_e} is a function of the maximization (22). At a particular point in time, it is possible for this function to have more than one local maximum. For $\theta = 5$, Figure 11 shows that this sum around the time that the worst case paramter begins to follow the estimated parameter has two peaks. At a certain point in time, the peak which more closely follows the parameter estimate begins to dominate as the decreasing variance causes perturbations from the estimated value to be weighted more heavily. At this point in time, the worst case parameter jumps to a value corresponding to this peak. Then, as this peak begins dominating the other peak, the controller begins to use the elevator deflection more heavily.

6 Conclusions

The results presented in this paper demonstrate the performance of a robust adaptive compensator based on disturbance attenuation. These results show that as the disturbance attenuation bound is lowered, the compensator tends to rely more upon controls whose effects are know with more certainty until estimates of uncertain coefficients are known with enough confidence to be used effectively. The most dramatic improvements in performance occur when the initial estimates of the unknown parameters are farthest from their true values.

For systems which contain a great deal of parameter uncertainty in some or all of the coefficients, such as flight control systems, this type of compensator can prove to be very useful. Without any a priori restrictions on the structure of the compensator, the disturbance attenuation approach results in a design which is both robust in that it tends to hedge against uncertain states and parameters, and adaptive in that it uses the measurement history to update its knowledge of the unknown parameters. This type of design is particularly useful for systems, such as the flight control system examined in this paper, in which parameters may vary in magnitude and/or sign over varying conditions.

By using this robust adaptive compensator, the controller not only has the ability to update its information of the system model, but is designed in such a way that it chooses its control based on how well this model, or parameters within the model, is known. As more information becomes available to the controller, the parameters within the model become more well known, and the controller is able to use this increased certainty in the system model to utilize controls which are most affected by the uncertain coefficients with more confidence. By using the controls which are known with the greatest certainty, the overall performance of the system can then be improved.

7 Acknowledgements

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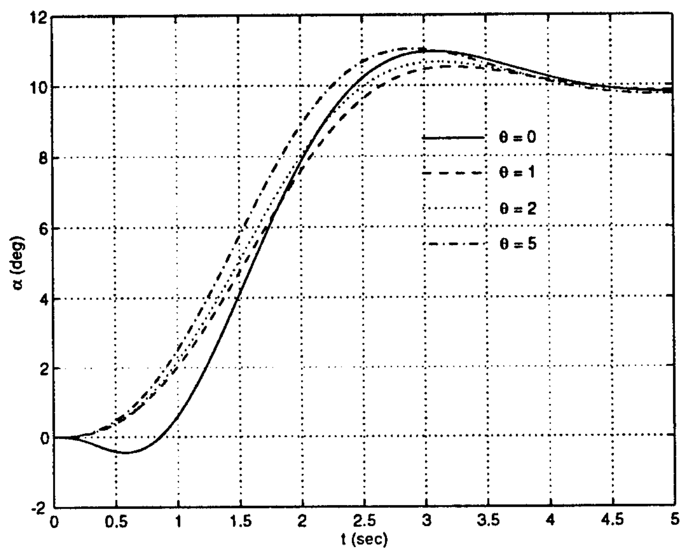

Figure 1: Angle of Attack Step Response

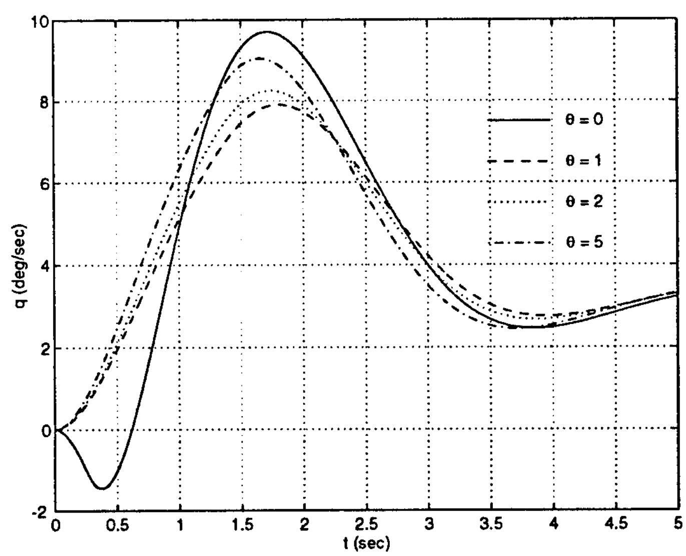

Figure 2: Pitch Rate Response

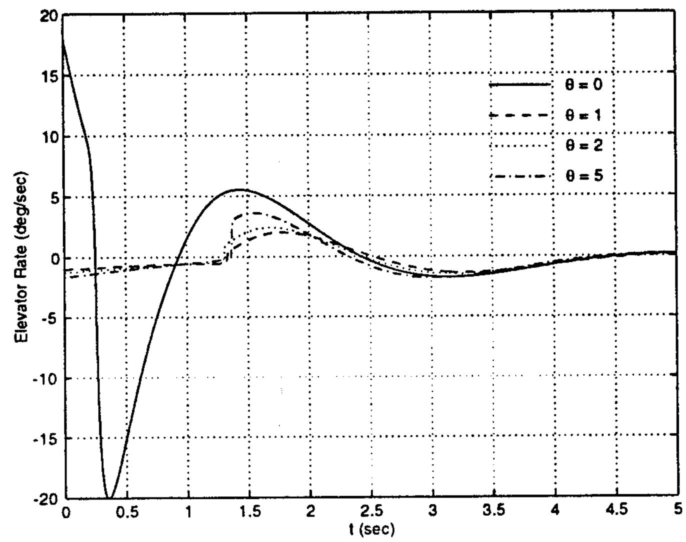

Figure 3: Elevator Command Rate, $\dot{\delta}_e$

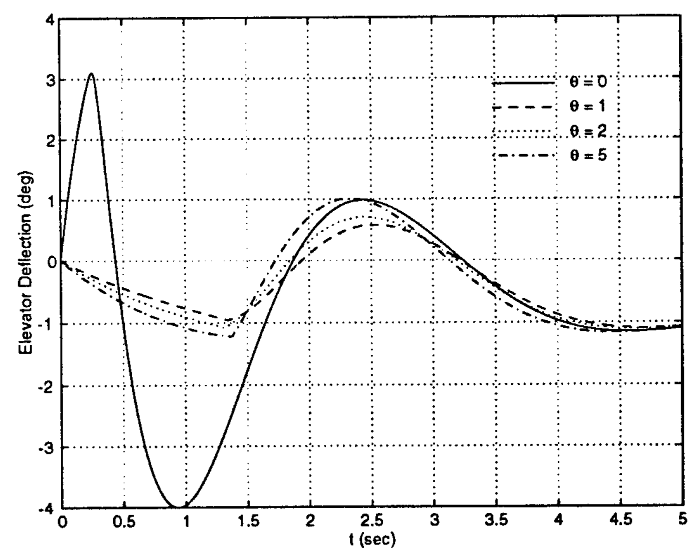

Figure 4: Elevator Command, δ_e

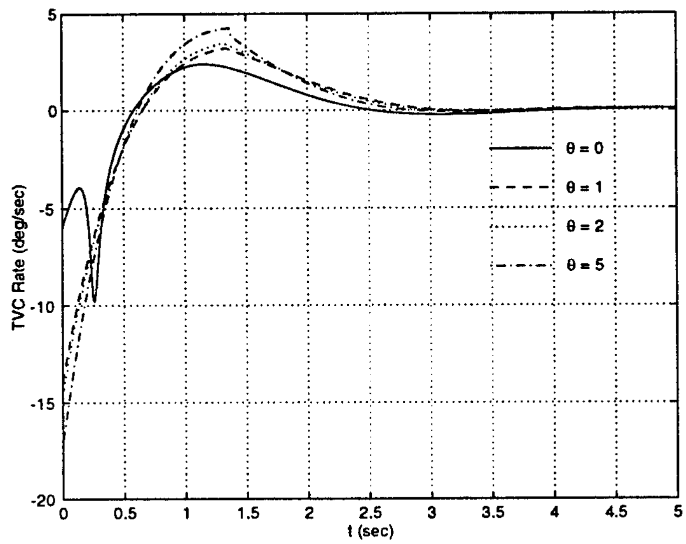

Figure 5: Thrust Vector Command Rate, $\dot{\delta}_{TVC}$

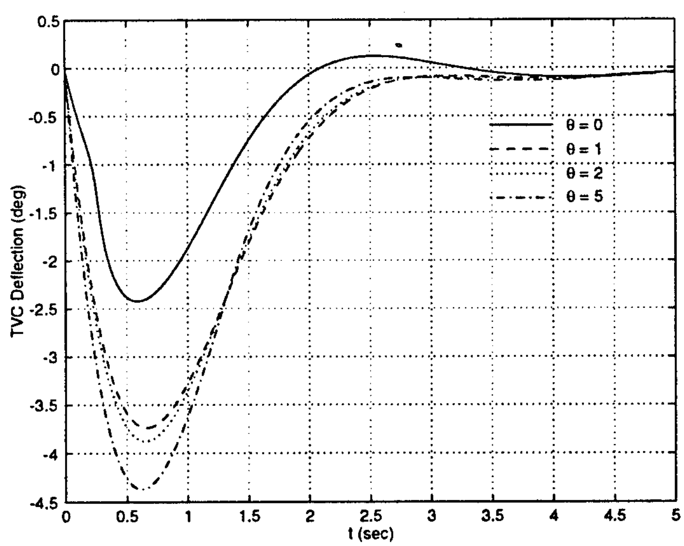

Figure 6: Thrust Vector Command, δ_{TVC}

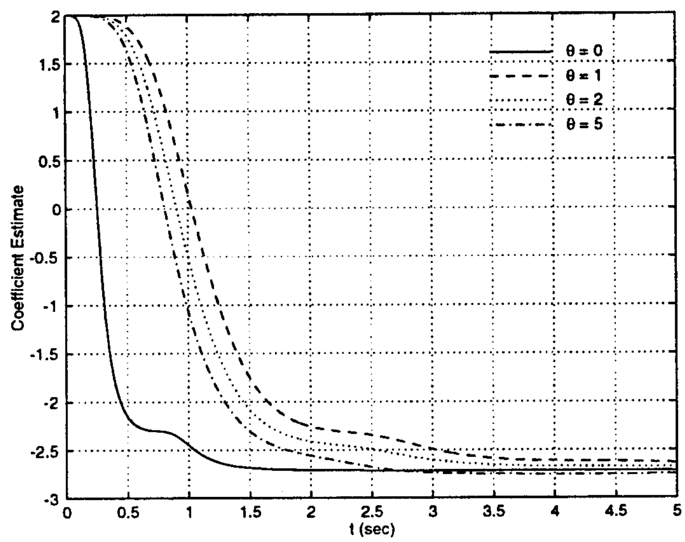

Figure 7: Estimated Parameter Value, $\hat{M}_{\delta_e}$

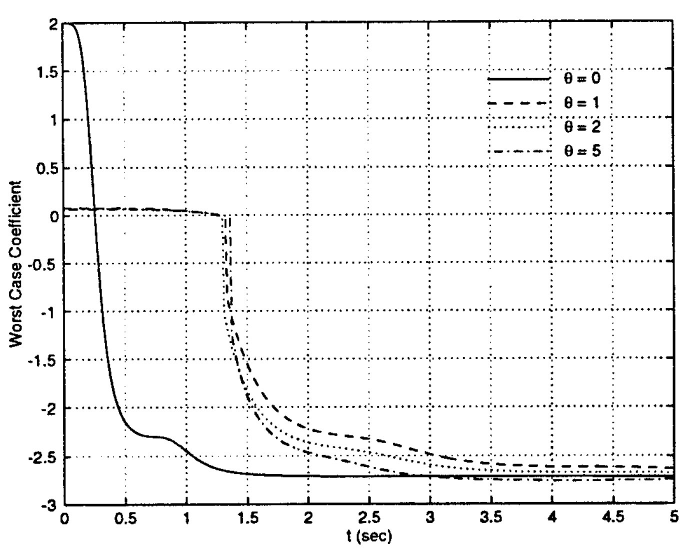

Figure 8: Worst Case Parameter, $M^*_{\delta_e}$

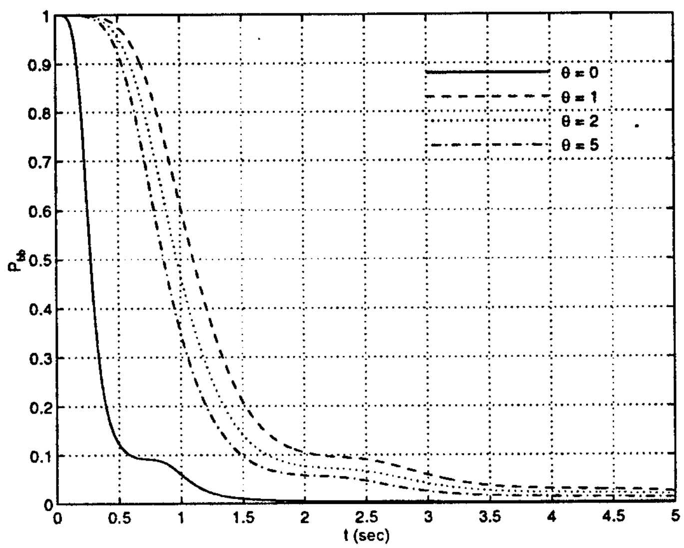

Figure 9: Parameter Variance, $P_{M_{\delta_e}}$

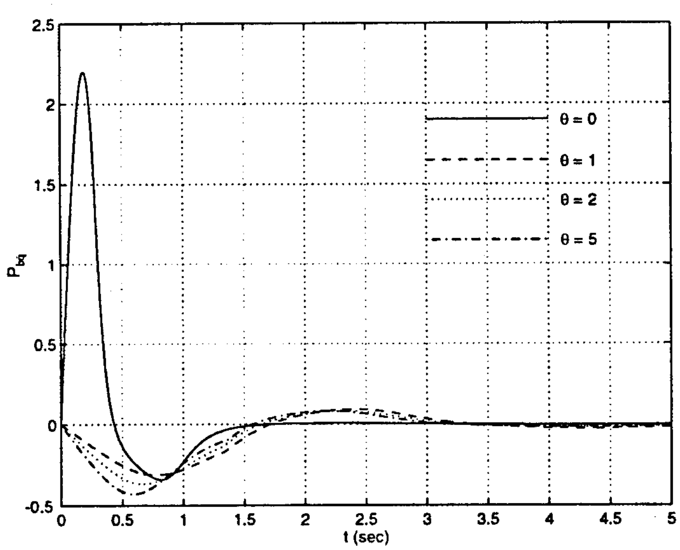

Figure 10: Parameter/Pitch Acceleration Cross Covariance, $P_{M_{\delta_e}\dot{q}}$

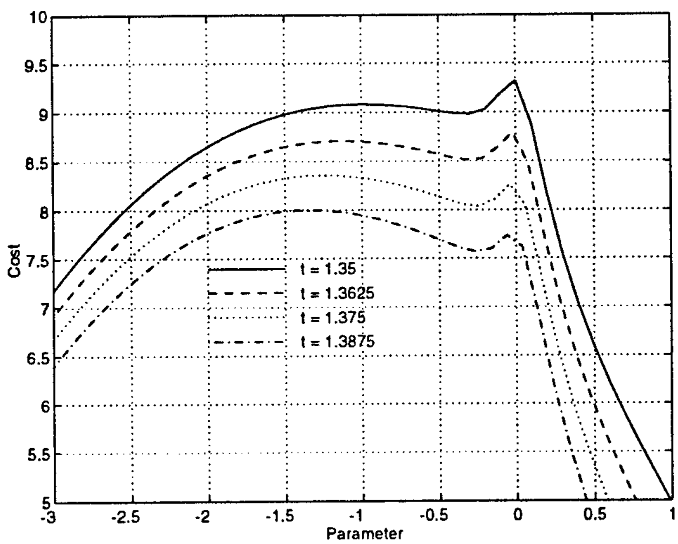

Figure 11: Cost, $\left[\frac{1}{2}x^T\Pi x - \frac{1}{2\theta}e^T S e\right]$ As a Function of M_{δ_e}

Appendix D

NONLINEAR APPROXIMATE GAME-THEORETIC ESTIMATION AND CONTROL

Approximate Nonlinear Game-Theoretic Estimation and Control

Jinsheng Jang and Jason L. Speyer

University of California, Los Angeles

Abstract

For a class of linear dynamical systems with nonlinear perturbation, an approximate optimal estimation scheme is derived based on a deterministic game-theoretic criterion. Using the calculus of variation approach, the process disturbance and initial state vector are first maximized. The resulting optimality condition is then expanded with respect to a small parameter to solve for the worst case state and the Lagrange multiplier term by term. Subsequently, the approximate optimal estimator is derived by minimizing a series of cost criterions over its corresponding state estimate vectors. The estimator Riccati differential equations (RDE) necessary for the first and higher order correction terms are shown to be the same as in the zeroth-order case. Then, the infinite order approximate minimax estimator is shown to be disturbance attenuating. In addition, the N-th order approximate minimax estimator is proved to achieve disturbance attenuation with a higher threshold proportional to the $N+1$ power of the expansion parameter. Therefore, as more and more terms are added, the N-th order threshold moves closer and closer to the infinite case. For a similar class of linear dynamical system with similar perturbation, an approximate game-theoretic measurement feedback controller is obtained. As in the estimation problem, first a maximization problem with respect to the disturbance and initial state vector is performed. Then, a minimax deterministic game is considered where the measurement noise is acting as a maximizing player and the control is treated as a minimizing player. The analytical form of the zeroth-order term for the approximate controller resembles the linear solution. However, the first and higher order correction terms of the controller require two estimator where one of the estimator is almost the same as in the pure estimation case. The other estimator is derived from a two-point boundary value problem associated with the control minimization sub-problem. For the zeroth-order case, the two RDEs for the control problem are the same as in the linear case. Although the estimator RDE remains the same as the order of the problem progresses, the controller RDE for the first and higher order problem is different from the zeroth-order case. Finally, the approximate game-theoretic controller is proved to have disturbance attenuation property when infinite order terms are used. Furthermore, this property is kept when only finite correction terms are used for the approximate controller using an increased threshold value proportional to the $N+1$ power of the expansion parameter. As expected, this threshold moves closer and closer to the infinite order threshold as more and more correction terms

are utilized in the approximate controller and recovers the original threshold when infinite correction terms are incorporated.

I. Introduction

In the literature, the research in the area of the deterministic approximate optimal guidance has been well documented [1, 2, 3]. Two approaches, namely the Hamilton-Jacobi-Bellman equation expansion technique [1] and the calculus of variation approach [2, 3], are related to our works here. Surprisingly, very few reports on the equally important approximate optimal estimation problem using the above two deterministic approaches have been investigated. In addition, the disturbance attenuation property of the approximate guidance scheme as derived in [1, 2, 3] has not been studied. To the authors' knowledge, only the stochastic approximate optimal estimation using also a power series expansion technique is found in [4]. In [4], for a specific polynomial nonlinearity of order 3, an approximate estimation scheme is derived by indirectly calculating the approximate conditional density function using perturbation method. Their results are only valid for a scalar system. Furthermore, in [4], explicit formula to determine the expansion terms of the conditional density is obtained only for a very simple example. Recently, in [5], a deterministic game problem as here is formulated for a singularly perturbed dynamical system. Only the linear dynamical system is treated in [5].

In this work, using a regular perturbation approach, for a class of linear dynamical system with nonlinear perturbations in both system dynamics and measurement equations, an approximate estimation scheme is derived. A calculus of variation approach [6] is used to derive the estimator based on a deterministic game-theoretic performance index. First, by expanding a nonlinear, two-point boundary value problem with respect to the small parameter, a series of linear, solvable two-point boundary value problems are obtained and solved for the worst case state and Lagrange multiplier vectors term by term. Next, setting the small parameter to zero, the zeroth-order solution for the approximate filter is shown to be the linear H_∞ estimator as derived in [7]. Furthermore, first and the higher order correction terms for the approximate game-theoretic estimator are derived via the minimization of a series of performance indexes with respect to its corresponding state estimate correction terms. The performance indexes are results of the power series expansion of the original performance index with respect to the same small parameter. Since only the filtering problem is of interest here, the resulting estimation equation is simplified by substituting the boundary condition of the Lagrange Multiplier into its associated optimality condition. The RDEs for the higher order correction terms are derived to be the same as the RDE in the linear H_∞ problem.

Subsequently, the disturbance attenuation problem as defined in section II is shown to be solved by this infinite-order approximate game-theoretic filter. Furthermore, the N-th order disturbance attenuation problem is proved to be solved by the N-th order approximate game-theoretic filter which is truncated from the infinite order filter. Particularly, the disturbance attenuation threshold for the approximate game-theoretic filter of arbitrary order is shown to be upper bounded by twice the original threshold and the increase in the threshold is proportional to the $N + 1$ power of the expansion parameter. Therefore, as the order of the estimator increases, the threshold decreases and reduces to the original threshold as the order approches infinity.

Next, the approximate game-theoretic control is formulated and derived by solving a disturbance attenuation problem with a regular perturbation approach bu using the calculus of variations. The derivations are separated into two parts. First, a maximization problem is solved with respect to the disturbance and initial state vectors. Following that, the first-order necessary conditions are derived. Similarly as in the estimation case, a nonlinear two-point boundary value problem is formed based on the original system dynamics and the first-order optimality conditions. Naturally, both this nonlinear two-point boundary value problem and the cost function are expanded using a combination of power series expansion and Taylor's series expansion. Secondly, a series of linear minimax deterministic game problems with respect to the expansion terms of the cost function are solved sequentially again using a calculus of variation approach. In each problem, the first order necessary condition is derived, then another two-point boundary value problem is formed based on the necessary condition.

The zeroth-order solution of the approximate controller resembles the linear case as obtained in [9]. The first order and higher order correction terms of the controller require two estimator. The first estimator is almost the same in the pure estimation case. The second estimator is derived from the two-point boundary value problem from the control minimization part of the problem solution. For each expansion term, the controller requires the solutions of two RDEs, namely the estimator RDE and the controller RDE. The estimator RDe remains intact as the order of the correction terms progresses. However the controller RDE for the first and higher order case is different from the zeroth-order solution. This is due to the fact that the measurement vector is a given sequence and thus not expanded. Therefore, the measurement vector is treated as a zeroth-order term only. As expected, similar disturbance attenuation results are proved.

The present paper is organized as follows. In section II, the disturbance attenuation problem for the approximate game-theoretic estimation is folumated. The problem is then solved using a game-theoretic approach. Using a caculus of variation technique, the first order necessary condition is derived for the maximization problem. In section III, the zeroth-order solution is obtained solving a minimization problem with respect to the zeroth-order term of the cost function. The first order and higher order correction terms for the approximate estimator are derived in section IV. Section V presents the results of the disturbance attenuation property of the estimator. Finally, the approximate game-theoretic controller for a output feedback control problem is derived in section VI.

II. Problem Formulation

Consider a linear dynamical system with nonlinear perturbation

$$\dot{x} = Ax + \epsilon g(x) + \Gamma w \tag{1}$$

and the measurement equation with nonlinear perturbation as

$$z = Hx + \epsilon h(x) + v \tag{2}$$

where $x \epsilon R^n$, $w \epsilon R^d$, and $v \epsilon R^m$ are the state, process disturbance, and the measurement noise vectors, respectively. In this paper, $g(x)$ and $h(x)$ are assumed to be continuous and infinitely differentiable. To start, denote a disturbance attenuation function as

$$D_{af}(\hat{x}, w, x(0)) \equiv \frac{\int_0^t \|x - \hat{x}\|_Q^2 d\tau}{\|x(0) - \hat{x}(0)\|_{P_0^{-1}}^2 + \int_0^t \|w\|_{W^{-1}}^2 + \|z - Hx - \epsilon h(x)\|_{V^{-1}}^2 d\tau} \tag{3}$$

where $\hat{x}$ is the state estimate vector. The numerator and the denominator of the disturbance attenuation function are measures of the output and input energy using quadratic norm. Based on (3), the disturbance attenuation problem is defined as to find an optimal estimator $\hat{x}^*(\cdot)$ such that

$$D_{af}(\hat{x}^*, w, x(0)) \leq \theta, \;\; \theta > 0 \tag{4}$$

for all $w \epsilon L_2[0, t]$ and $x(0) \epsilon R^n$ such that $(w(\tau), v(\tau)) \neq 0$ for all $\tau \epsilon [0, t]$ and $x(0) \neq \hat{x}(0)$. In (4), θ is called the disturbance attenuation threshold, $\hat{x}^*(\cdot)$ is defined as a causal mapping as

$$\hat{x}^*(\tau) \equiv \Psi[Z(\tau)] \tag{5}$$

where $Z(\tau) = \{z(s) | 0 \leq s \leq \tau\}$ is the measurement history. Intuitively, θ can be interpreted as a upper bound of a transfer function measuring the effect of the input norm upon the output error norm.

To solve the disturbance attenuation problem as defined in (4), we adopt a minimax approach. A seperable performance index is obtained from (4) as

$$\begin{aligned} J(\hat{x}; w, x(0)) \;\; &\equiv \;\; -\frac{\theta}{2}\|x(0) - \hat{x}(0)\|_{P_0^{-1}}^2 + \int_0^t \frac{1}{2}\|x - \hat{x}\|_Q^2 - \frac{\theta}{2}[\|w\|_{W^{-1}}^2 \\ &+ \|z - Hx - \epsilon h\|_{V^{-1}}^2] d\tau \end{aligned} \tag{6}$$

The problem is to find an estimation scheme based on the optimization of the performance index in (6) for the system described as in (1) and (2). Thus, consider the following deterministic game

$$min_{\hat{x}} max_{x(0),w} \; J, \tag{7}$$

where $\hat{x}(\cdot)$ is trying to minimize J while the adversaries, $x(0)$ and $w(\cdot)$, are intending to maximize J. J is as chosen in (6) subject to constraints (1) and (2) which is explicitly introduced into the cost J using a Lagrange Multiplier λ. Note that the maximization of the measurement noise is implicit through the constraint (2) since the measurement vector z is given and the maximization of the initial state vector $x(0)$ and the process disturbance w leaves no more freedom to the measurement noise.

The maximization procedure is tackled first. For our conviences, denote

$$J^*(\hat{x}; x^*(0), w^*) = max_{x(0),w} \; J. \tag{8}$$

From the calculus of variation approach [5], the first order necessary conditions of optimality are easily obtained as

$$\begin{aligned} &\lambda^T(0) = [x(0) - \hat{x}(0)]^T P_0^{-1}, \;\; \lambda^T(t) = 0, \\ &\dot{\lambda}^T + \lambda^T \epsilon g_x + \lambda^T A + (z - Hx - \epsilon h)^T V^{-1}(H + \epsilon h_x) + \theta^{-1}(x - \hat{x})^T Q = 0 \\ &-w^T W^{-1} + \lambda^T \Gamma = 0 \end{aligned} \tag{9}$$

where the partial derivativve matrices are defined as $h_x \equiv \frac{\partial h(x)}{\partial x}$ and $g_x \equiv \frac{\partial g(x)}{\partial x}$. From (9) and (1), a nonlinear two point boundary value problem is formed as

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} &= \begin{bmatrix} A & \Gamma W \Gamma^T \\ (H + \epsilon h_x)^T V^{-1} H - \theta^{-1} Q & -A^T - \epsilon (g_x)^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \\ &+ \begin{bmatrix} \epsilon g \\ -(H + \epsilon h_x)^T V^{-1}(z - \epsilon h) + \theta^{-1} Q \hat{x} \end{bmatrix}, \;\; x(0) = \hat{x}(0) + P_0 \lambda(0), \;\; \lambda(t) = 0 \end{aligned} \tag{10}$$

II.1 An Ordinary Perturbation Approach to the Solution of the Disturbation Problem

In general, it is impossible to solve for the worst case x and λ analytically from (10). Therefore, the worst case x and λ are determined approximately by expanding (10) in a power series [2]. Note that z is a given data sequence and is not expanded below. To start the expansion, let

$$x = \sum_{j=0}^{\infty} x_j \epsilon^j, \;\; \lambda = \sum_{j=0}^{\infty} \lambda_j \epsilon^j, \;\; \hat{x} = \sum_{j=0}^{\infty} \hat{x}_j \epsilon^j. \tag{11}$$

Subsequently, g, g_x, h, and h_x are expanded using Taylor's series expansion with respect to x_0 as

$$
\begin{aligned}
g(x) &= \bar{g} + \bar{g}_x(x - x_0) + \frac{1}{2}(x - x_0)^T \bar{g}_{xx}(x - x_0) + \cdots \\
&= g_0 + \epsilon g_1 + \epsilon^2 g_2 + \cdots && (12) \\
g_x(x) &= g_{x_0} + \epsilon g_{x_1} + \epsilon^2 g_{x_2} + \cdots && (13)
\end{aligned}
$$

where

$$
\begin{aligned}
g_0 &\equiv \bar{g} \equiv g(x)|_{x=x_0} \\
g_1 &\equiv \bar{g}_x x_1 \equiv \frac{\partial g(x)}{\partial x}|_{x=x_0} x_1 \\
g_2 &\equiv \bar{g}_x x_2 + \frac{1}{2} x_1^T \bar{g}_{xx} x_1 \equiv \frac{\partial g(x)}{\partial x}|_{x=x_0} x_2 + \frac{1}{2} x_1^T \frac{\partial^2 g(x)}{\partial x^2}|_{x=x_0} x_1 && (14)
\end{aligned}
$$

$$
\begin{aligned}
g_{x_0} &\equiv \bar{g}_x = \frac{\partial g}{\partial x}|_{x=x_0} \\
g_{x_1} &\equiv \bar{g}_{xx} x_1 = \frac{\partial^2 g}{\partial x^2}|_{x=x_0} x_1 \\
g_{x_2} &\equiv \bar{g}_{xx} x_2 + \frac{1}{2} x_1^T \bar{g}_{xxx} x_1 && (15)
\end{aligned}
$$

Note that $\bar{g}_{xx}, \bar{g}_{xxx}$, etc., are tensors. For example, for our conveniences, denote

$$
(x - x_0)^T \bar{g}_{xx}(x - x_0) = \begin{bmatrix} (x - x_0)^T \bar{g}_{1,xx} \\ \cdot \\ \cdot \\ \cdot \\ (x - x_0)^T \bar{g}_{n,xx} \end{bmatrix} (x - x_0) \qquad (16)
$$

as a column vector instead of using tensor notation. Similarly, $h(x)$ and $h_x(x)$ can be expanded. The related notation is self-explanatory. Equation (10) is then expanded using (11), (12), and (13). After equating the coefficients of like powers of ϵ, a series of linear, two-point boundary value problems are obtained as

$$
\begin{aligned}
\epsilon^0 : \begin{bmatrix} \dot{x}_0 \\ \dot{\lambda}_0 \end{bmatrix} &= \begin{bmatrix} A & \Gamma W \Gamma^T \\ (H^T V^{-1} H - \theta^{-1} Q) & -A^T \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} + \begin{bmatrix} 0 \\ -H^T V^{-1} z + \theta^{-1} Q \hat{x}_0 \end{bmatrix} \\
x_0(0) &= \hat{x}(0) + P_0 \lambda_0(0);\ \lambda_0(t) = 0 && (17) \\
\epsilon^1 : \begin{bmatrix} \dot{x}_1 \\ \dot{\lambda}_1 \end{bmatrix} &= \begin{bmatrix} A & \Gamma W \Gamma^T \\ (H^T V^{-1} H - \theta^{-1} Q) & -A^T \end{bmatrix} \begin{bmatrix} x_1 \\ \lambda_1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+\begin{bmatrix} g_0 \\ -h_{x_0}^T V^{-1}(z-Hx_0)+H^TV^{-1}h_0-g_{x_0}^T\lambda_0+\theta^{-1}Q\hat{x}_1 \end{bmatrix}, \\
x_1(0) &= P_0\lambda_1(0);\ \lambda_1(t)=0 \qquad (18) \\
\epsilon^2:\begin{bmatrix} \dot{x}_2 \\ \dot{\lambda}_2 \end{bmatrix} &= \begin{bmatrix} A & \Gamma W\Gamma^T \\ (H^TV^{-1}H-\theta^{-1}Q & -A^T \end{bmatrix}\begin{bmatrix} x_2 \\ \lambda_2 \end{bmatrix} \\
&+\begin{bmatrix} g_1 \\ g_f \end{bmatrix} \\
x_2(0) &= P_0\lambda_2(0);\ \lambda_2(t)=0 \qquad (19)
\end{aligned}
$$

where

$$
\begin{aligned}
g_f \equiv & -h_{x_1}^T V^{-1}(z-Hx_0)+h_{x_0}^TV^{-1}(h_0+Hx_1)-g_{x_1}^T\lambda_0-g_{x_0}^T\lambda_1 \\
&+H^TV^{-1}h_1+\theta^{-1}Q\hat{x}_2. \qquad (20)
\end{aligned}
$$

Let x_j^* and λ_j^* be the solutions to the corresponding problems in (17)-(19). Denote

$$
x^*=\sum_{j=0}^{\infty}x_j^*\epsilon^j,\ \ \lambda^*=\sum_{j=0}^{\infty}\lambda_j^*\epsilon^j. \qquad (21)
$$

Similarly,

$$
g(x^*)=\sum_{j=0}^{\infty}g_j^*\epsilon^j,\ \ g_{x^*}(x^*)=\sum_{j=0}^{\infty}g_{x_j}^*\epsilon^j,\ \ h(x^*)=\sum_{j=0}^{\infty}h_j^*\epsilon^j,\ \ h_{x^*}(x^*)=\sum_{j=0}^{\infty}h_{x_j}^*\epsilon^j \qquad (22)
$$

Furthermore, J^* is expanded as

$$
\begin{aligned}
J^* = & -\frac{\theta}{2}\|\sum_{j=0}^{\infty}\lambda_j^*(0)\epsilon^j\|_{P_0}^2+\int_0^t\frac{1}{2}\|(\sum_{j=0}^{\infty}x_j^*\epsilon^j)-(\sum_{j=0}^{\infty}\hat{x}_j\epsilon^j)\|_Q^2 \\
& -\frac{\theta}{2}\|\sum_{j=0}^{\infty}\lambda_j^*\epsilon^j\|_{\Gamma W\Gamma^T}^2-\frac{\theta}{2}\|z-H(\sum_{j=0}^{\infty}x_j^*\epsilon^j)-\epsilon(\sum_{j=0}^{\infty}h_j^*\epsilon^j)\|_{V^{-1}}^2d\tau. \qquad (23)
\end{aligned}
$$

Denote $J^*=\sum_{i=0}^{\infty}J_i\epsilon^i$. Thus

$$
\begin{aligned}
J_0 &= -\frac{\theta}{2}\|\lambda_0^*(0)\|_{P_0}^2+\int_0^t\frac{1}{2}\|x_0^*-\hat{x}_0\|_Q^2-\frac{\theta}{2}\|\lambda_0^*\|_{\Gamma W\Gamma^T}^2-\frac{\theta}{2}\|z-Hx_0^*\|_{V^{-1}}^2d\tau \qquad (24) \\
J_1 &= -\theta[\lambda_0^*(0)]^TP_0\lambda_1^*(0)+\int_0^t(x_0^*-\hat{x}_0)^TQ(x_1^*-\hat{x}_1)-\theta[\lambda_0^*]^T\Gamma W\Gamma^T\lambda_1^*
\end{aligned}
$$

$$
\begin{aligned}
&-\theta(z-Hx_0^*)^T V^{-1}(-Hx_1^*-h_0^*)d\tau && (25)\\
J_2 \;=\;& -\theta[\lambda_0^*(0)]^T P_0\lambda_2^*(0)-\frac{\theta}{2}\|\lambda_1^*(0)\|_{P_0}^2+\int_0^t (x_0^*-\hat{x}_0)^T Q(x_2^*-\hat{x}_2)\\
&+\frac{1}{2}\|x_1^*-\hat{x}_1\|_Q^2-\theta[\lambda_0^*]^T\Gamma W\Gamma^T\lambda_2^*-\frac{\theta}{2}\|\lambda_1^*\|_{\Gamma W\Gamma^T}^2\\
&-\theta(z-Hx_0^*)^T V^{-1}(-Hx_2^*-h_1^*)-\frac{\theta}{2}\|Hx_1^*+h_0^*\|_{V^{-1}}^2 d\tau && (26)
\end{aligned}
$$

Note that the ϵ^0 terms in (17) and (24) are exactly the same as in the linear case [6]. Thus, zeroth-order solution is indeed the linear H_∞ estimator which will be derived in section III by minimizing J_0 with respect to $\hat{x}_0$ subject to its associated constraints. As shown in section IV, the first order correction term for the estimator will be obtained by minimizing J_2 with repect to the first order correction term of the state estimate, namely, $\hat{x}_1$ subject to its associated constraints. Based on this philosophy of the perturbation method, the higher order correction terms of the estimator are to be determined in section IV.

III. Zeroth-Order Solution - the Linear H_∞ Estimator

To derive the zeroth-order solution, consider the linear, two-point boundary value problem as in (17). Assume

$$
x_0^* = x_{c_0} + P\lambda_0^* \tag{27}
$$

where x_{c_0} is an intermediate variable and P is the Riccati variable, both of which are to be evaluated later. Differentiation of the above equation and substitution of both sides of (17) yields

$$
\begin{aligned}
&[AP+\Gamma W\Gamma^T-\dot{P}-P(H^TV^{-1}H-\theta^{-1}Q)P+PA^T]\lambda_0^*\\
=\;& -Ax_{c_0}+\dot{x}_{c_0}+P(H^TV^{-1}H-\theta^{-1}Q)x_{c_0}-PH^TV^{-1}z+\theta^{-1}PQ\hat{x}_0
\end{aligned}
$$

Since λ_0^* is arbitrary, the choices of

$$
\begin{aligned}
\dot{x}_{c_0} &= Ax_{c_0}+PH^TV^{-1}(z-Hx_{c_0})+\theta^{-1}PQ(x_{c_0}-\hat{x}_0),\, x_c(0)=\hat{x}(0) && (28)\\
\dot{P} &= AP+PA^T+\Gamma W\Gamma^T-P(H^TV^{-1}H-\theta^{-1}Q)P,\;\; P(0)=P_0. && (29)
\end{aligned}
$$

guarantee the satisfaction of (27). Add the zero identity

$$
\frac{\theta}{2}\|\lambda_0^*(0)\|_{P_0}^2-\frac{\theta}{2}\|\lambda_0^*(t)\|_{P(t)}^2+\frac{\theta}{2}\int_0^t\frac{d}{d\tau}[(\lambda_0^*)^TP\lambda_0^*]d\tau=0
$$

to J_0 as in (24). Thus

$$J_0 = \frac{\theta}{2}\int_0^t \theta^{-1}\|x_{c_0} - \hat{x}_0\|_Q^2 - \|z - Hx_{c_0}\|_{V^{-1}}^2 d\tau. \tag{30}$$

Now consider the minimization problem

$$min_{\hat{x}_0} J_0(\hat{x}_0, x_{c_0})$$

subject to constraints (28) and (29). Note that the advantage of perturbation method is exploited here to obtain the optimal $\hat{x}_0$ by neglecting first and higher order terms of the expansion of the cost function. Thus, using the Lagrange Multiplier technique, the first order optimality condition is obtained as

$$\begin{aligned} 0 &= Q(x_{c_0} - \hat{x}_0) + QP\zeta & (31)\\ \dot{\zeta} &= -A^T\zeta + H^TV^{-1}HP\zeta - \theta^{-1}QP\zeta - \theta^{-1}Q(x_{c_0} - \hat{x}_0) - H^TV^{-1}(z - Hx_{c_0}), \\ \zeta(t) &= 0 & (32)\end{aligned}$$

The result of substitution of the boundary condition $\zeta(t) = 0$ into (31) is subsequently substituted into (28). Thus

$$\dot{\hat{x}}_0^* = A\hat{x}_0^* + PH^TV^{-1}[z - H\hat{x}_0^*], \;\; \hat{x}_0^*(0) = \hat{x}(0), \tag{33}$$

where P is determined from (29). Note that (33) is obtained using the fact that the current time t is arbitrary. Equations (33) and (29) form the linear H_∞ estimator. Note that the worst case strategies for x_0 and λ_0 are denoted as x_0^* and λ_0^* which are calculated from

$$\begin{aligned} \begin{bmatrix} \dot{x}_0^* \\ \dot{\lambda}_0^* \end{bmatrix} &= \begin{bmatrix} A & \Gamma W \Gamma^T \\ (H^TV^{-1}H - \theta^{-1}Q) & -A^T \end{bmatrix} \begin{bmatrix} x_0^* \\ \lambda_0^* \end{bmatrix} + \begin{bmatrix} 0 \\ -H^TV^{-1}z + \theta^{-1}Q\hat{x}_0 \end{bmatrix} \\ x_0^*(0) &= \hat{x}(0) + P_0\lambda_0^*(0); \; \lambda_0^*(t) = 0 \end{aligned} \tag{34}$$

Using $\hat{x}_0^*$, the minimax strategies of x_0 and λ_0 are denoted as x_0^m and λ_0^m which are determined from

$$\begin{aligned} \begin{bmatrix} \dot{x}_0^m \\ \dot{\lambda}_0^m \end{bmatrix} &= \begin{bmatrix} A & \Gamma W \Gamma^T \\ (H^TV^{-1}H - \theta^{-1}Q) & -A^T \end{bmatrix} \begin{bmatrix} x_0^m \\ \lambda_0^m \end{bmatrix} + \begin{bmatrix} 0 \\ -H^TV^{-1}z + \theta^{-1}Q\hat{x}_0^* \end{bmatrix} \\ x_0^m(0) &= \hat{x}(0) + P_0\lambda_0^m(0); \; \lambda_0^m(t) = 0. \end{aligned} \tag{35}$$

IV. First and Higher Order Correction Terms

IV.1 First Order Correction Term

To obtain the first order correction term for the approximate estimation scheme, we first solve for worst case x_1 and λ_1 from the ϵ^1 term in (18). After that, the correction term is obtained from the minimization of $\epsilon^2 J_2$ with respect to $\hat{x}_1$ subject to its associated constraints. Note that the first order necessary condition of the minimization problem of J_1 with respect to $\hat{x}_1$ is equivalent to the first order necessary condition of the minimization problem of J_0 with respect to $\hat{x}_0$. Using the sweep method [5], assume

$$x_1^* = x_{c_1} + P\lambda_1^*. \tag{36}$$

Differentiation of both sides of (36) yields

$$\begin{aligned} & [-\dot{P} + AP + PA^T + \Gamma W \Gamma^T - P(H^T V^{-1} H - \theta^{-1} Q)P]\lambda_1^* \\ = \ & \dot{x}_{c_1} - Ax_{c_1} + PH^T V^{-1}(h_0^* + Hx_{c_1}) - \theta^{-1} PQ(x_{c_1} - \hat{x}_1) - P(h_{x_0}^*)^T V^{-1}(z - Hx_0^*) \\ & -g_0^* - P(g_{x_0}^*)^T \lambda_0^* \end{aligned}$$

where (18) is used. Choosing

$$\begin{aligned} \dot{x}_{c_1} &= Ax_{c_1} - PH^T V^{-1}(h_0^* + Hx_{c_1}) + \theta^{-1} PQ(x_{c_1} - \hat{x}_1) + P(h_{x_0}^*)^T V^{-1}(z - Hx_0^*) \\ & \quad + g_0^* + P(g_{x_0}^*)^T \lambda_0^*, \ x_{c_1}(0) = 0 && (37) \\ \dot{P} &= AP + PA^T + \Gamma W \Gamma^T - P(H^T V^{-1} H - \theta^{-1} Q)P, \ \ P(0) = P_0 && (38) \end{aligned}$$

renders (36) an identity and the boundary condition of (18) satisfied. Before we proceed further, add the zero identity

$$\begin{aligned} 0 \ &= \ \theta[\lambda_0^*(0)]^T P_0 \lambda_2^*(0) - \theta[\lambda_0^*(t)]^T P(t)\lambda_2^*(t) + \frac{\theta}{2}\|\lambda_1^*(0)\|_{P_0}^2 - \frac{\theta}{2}\|\lambda_1^*(t)\|_{P(t)}^2 \\ & \quad + \theta \int_0^t \frac{d}{d\tau}[(\lambda_0^*)^T P \lambda_2^* + \frac{1}{2}(\lambda_1^*)^T P \lambda_1^*] d\tau && (39) \end{aligned}$$

to J_2. It follows that

$$J_2 = \theta \int_0^t I_2 d\tau \tag{40}$$

where

$$\begin{aligned} I_2 \ &= \ \frac{\theta^{-1}}{2}\|x_{c_1} - \hat{x}_1\|_Q^2 + \theta^{-1}(x_{c_0} - \hat{x}_0)^T Q(x_{c_2} - \hat{x}_2) - \frac{1}{2}\|Hx_{c_1} + h_0^*\|_{V^{-1}}^2 \\ & \quad -(\lambda_1^*)^T P[(h_{x_0}^*)^T V^{-1}(z - Hx_0^*) + (g_{x_0}^*)^T \lambda_0^*] + (z - Hx_{c_0})^T V^{-1}(Hx_{c_2} + h_1^*) \\ & \quad +(\lambda_0^*)^T P[(h_{x_0}^*)^T V^{-1}(Hx_1^* + h_0^*) - (h_{x_1}^*)^T V^{-1}(z - Hx_0^*) \\ & \quad -(g_{x_0}^*)^T \lambda_1^* - (g_{x_1}^*)^T \lambda_0^*]. && (41) \end{aligned}$$

after some algebra. Next, assuming that the zeroth-order term of the state estimate vector and the zeroth-order adversaries play optimally,

$$\begin{aligned}
J_2 &= J_2(\hat{x}_1, \hat{x}_2, x_{c_1}, x_{c_2}, x_1^*) \\
&= \theta \int_0^t + \frac{\theta^{-1}}{2} \|x_{c_1} - \hat{x}_1\|_Q^2 - \frac{1}{2}\|Hx_{c_1} + h_0^m\|_{V^{-1}}^2 - (\lambda_1^*)^T P[(h_{x_0}^m)^T V^{-1}(z - Hx_0^m) \\
&\quad +(g_{x_0}^m)^T \lambda_0^m] + (z - Hx_{c_0})^T V^{-1}(Hx_{c_2} + h_1^*) + (\lambda_0^m)^T P[(h_{x_0}^m)^T V^{-1}(Hx_1^* + h_0^m) \\
&\quad -(h_{x_1}^*)^T V^{-1}(z - Hx_0^m) - (g_{x_0}^m)^T \lambda_1^* - (g_{x_1}^*)^T \lambda_0^m] d\tau \qquad (42)
\end{aligned}$$

where the arguments of J_2 of are written out explicitly to avoid confusion. In (42), note that $x_{c_1}, x_1^*, \lambda_1^*$, and x_{c_2} are all driven by $\hat{x}_1$. However, from (36), x_{c_1}, x_1^*, and λ_1^* are not completely independent. Thus, it is sufficient to consider, say, only the constraint equations of x_{c_1} and x_1^*.

Now, when the first order disturbance and initial state vectors play their worst strategies, (18) becomes

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1^* \\ \dot{\lambda}_1^* \end{bmatrix} &= \begin{bmatrix} A & \Gamma W \Gamma^T \\ (H^T V^{-1} H - \theta^{-1} Q) & -A^T \end{bmatrix} \begin{bmatrix} x_1^* \\ \lambda_1^* \end{bmatrix} \\
&\quad + \begin{bmatrix} g_0^m \\ -h_{x_0^m}^T V^{-1}(z - Hx_0^m) + H^T V^{-1} h_0^m - g_{x_0^m}^T \lambda_0^m + \theta^{-1} Q \hat{x}_1 \end{bmatrix}, \\
x_1^*(0) &= P_0 \lambda_1^*(0);\ \lambda_1^*(t) = 0 \qquad (43)
\end{aligned}$$

where the zeroth-order term of the state estimate vector and the adversaries are assumed playing their minimax strategies. Substituting $\lambda_1^* = P^{-1}(x_1^* - x_{c_1})$ into the dynamics of x_1^* yields

$$\dot{x}_1^* = Ax_1^* + \Gamma W \Gamma^T P^{-1}(x_1^* - x_{c_1}) + g_0^m, x_1^*(0) = P_0 \lambda_1^*(0). \qquad (44)$$

After minimization of $\hat{x}_0$, (37) becomes

$$\begin{aligned}
\dot{x}_{c_1} &= Ax_{c_1} - PH^T V^{-1}(h_0^m + Hx_{c_1}) + \theta^{-1} PQ(x_{c_1} - \hat{x}_1) + P(h_{x_0}^m)^T V^{-1}(z - Hx_0^m) \\
&\quad + g_0^m + P(g_{x_0}^m)^T \lambda_0^m,\ x_{c_1}(0) = 0. \qquad (45)
\end{aligned}$$

The dynamics of x_{c_2} is similarly obtained as

$$\begin{aligned}
\dot{x}_{c_2} &= (A - PH^T V^{-1} H) x_{c_2} - \theta^{-1} PQ(x_{c_2} - \hat{x}_2) + g_1^* + P(h_{x_1}^*)^T V^{-1}(z - Hx_0^m) \\
&\quad - P(h_{x_0}^m)^T V^{-1}(h_0^m + Hx_1^*) + P(g_{x_1}^*)^T \lambda_0^m + P(g_{x_0}^m)^T \lambda_1^* - PH^T V^{-1} h_1^*, \\
x_{c_2}(0) &= 0 \qquad (46)
\end{aligned}$$

Next, consider the following minimization problem

$$min_{\hat{x}_1} J_2(\hat{x}_1, \hat{x}_2, x_{c_1}, x_{c_2}, x_1^*) \tag{47}$$

subject to the constraints in (46), (44), and (45). Introducing Lagrange Multiplier vectors β_1, σ_1, and γ_1 to combine the constraint equations as in (46), (44), and (45) with J_2 as in (42). Thus, the first order optimality conditions from the minimization problem (47) are

$$\begin{aligned}
0 &= Q(x_{c_1} - \hat{x}_1) + QP\gamma_1 && (48)\\
\dot{\gamma}_1 &= -A^T\gamma_1 + H^TV^{-1}HP\gamma_1 + P^{-1}g_{x_0}^m P\beta_1 + P^{-1}\Gamma W\Gamma^T\sigma_1 - P^{-1}g_{x_0}^m P\lambda_0^m \\
&\quad -(g_{x_0}^m)^T\lambda_0^m - (h_{x_0}^m)^TV^{-1}(z - Hx_0^m) + H^TV^{-1}(Hx_{c_1} + h_0^m), \;\; \gamma_1(t) = 0 && (49)\\
\dot{\sigma}_1 &= -A^T\sigma_1 - P^{-1}\Gamma W\Gamma^T\sigma_1 + (\frac{\partial h_1^*}{\partial x_1^*})^TV^{-1}HP\beta_1 - (\frac{\partial h_1^*}{\partial x_1^*})^TV^{-1}(z - Hx_{c_0}) \\
&\quad +[-(z - Hx_0^m)^TV^{-1}\frac{\partial h_{x_1}^*}{\partial x_1^*}P + H^TV^{-1}h_{x_0}^m P - (\lambda_0^m)^T\frac{\partial g_{x_1}^*}{\partial x_1^*}P - P^{-1}g_{x_0}^m P \\
&\quad -(\frac{\partial g_1^*}{\partial x_1^*})^T](\beta_1 - \lambda_0^m) + (h_{x_0}^m)^TV^{-1}(z - Hx_0^m), \;\; \sigma_1(0) = 0, && (50)\\
\dot{\beta}_1 &= -A^T\beta_1 + H^TV^{-1}HP\beta_1 - \theta^{-1}QP\beta_1 - H^TV^{-1}(z - Hx_{c_0}) - \theta^{-1}Q(x_{c_0} - \hat{x}_0^*), \\
\beta_1(t) &= 0, && (51)
\end{aligned}$$

where

$$\bar{g}_x^m = \frac{\partial g}{\partial x}|_{x=x_0^m};\; \bar{g}_{xx}^m = \frac{\partial^2 g}{\partial x^2}|_{x=x_0^m};\; \bar{h}_x^m = \frac{\partial h}{\partial x}|_{x=x_0^m};\; \bar{h}_{xx}^m = \frac{\partial^2 h}{\partial x^2}|_{x=x_0^m}. \tag{52}$$

In deriving (50), we have used the identity

$$[\frac{\partial g_1^*(x_0^m)}{\partial x_1^*}]^T = (g_{x_0}^m)^T. \tag{53}$$

Recall from (17),

$$\begin{aligned}
\dot{\lambda}_0^m &= -A^T\lambda_0^m + H^TV^{-1}HP\lambda_0^m - \theta^{-1}QP\lambda_0^m - H^TV^{-1}(z - Hx_{c_0}) \\
&\quad -\theta^{-1}Q(x_{c_0} - \hat{x}_0^*)\lambda_0^m(t) = 0 && (54)
\end{aligned}$$

Comparison of (51) and (54) reveals that for all τ

$$\lambda_0^m(\tau) = \beta_1(\tau). \tag{55}$$

After substitution of (55) into (50),

$$\dot{\sigma}_1 = -A^T\sigma_1 - P^{-1}\Gamma W\Gamma^T\sigma_1, \;\; \sigma_1(0) = 0, \tag{56}$$

where the zero identity

$$0 = HP\beta_1 - (z - Hx_{c_0}) + (z - Hx_0^m) \tag{57}$$

is used to derive (56). Thus, for all τ

$$\sigma_1(\tau) = 0. \tag{58}$$

Substituting (55) and (58) into (49) yields

$$\begin{aligned} \dot{\gamma}_1 \;\; = \;\; & -A^T\gamma_1 + H^TV^{-1}HP\gamma_1 - (g_{x_0}^m)^T\lambda_0^m - (h_{x_0}^m)^TV^{-1}(z - Hx_0^m) \\ & + H^TV^{-1}(Hx_{c_1} + h_0^m), \;\; \gamma_1(t) = 0. \end{aligned} \tag{59}$$

Substituting $\gamma_1(t) = 0$ into (48),

$$Q[x_{c_1}(t) - \hat{x}_1(t)] = 0 \tag{60}$$

Thus, the first order correction term of the optimal state estimate is obtained after substituting (60) into (45) as

$$\dot{\hat{x}}_1^* = A\hat{x}_1^* - PH^TV^{-1}[h_0^m + H\hat{x}_1^*(t)] + P(h_{x_0}^m)^TV^{-1}(z - Hx_0^m) + g_0^m + P(g_{x_0}^m)^T\lambda_0^m, \;\; \hat{x}_1^*(0) = 0. \tag{61}$$

and P is calculated as in (38). In (61), $h_0^m, h_{x_0}^m, g_0^m$, and $g_{x_0}^m$ are defined as in (12-15) when x_i plays its minimax strategy x_i^m. But from (27), $\lambda_0^* = P^{-1}(x_0^* - x_{c_0})$, then substitution into (17), we obtain

$$\dot{x}_0^* = (A + \Gamma W\Gamma^TP^{-1})x_0^* - \Gamma W\Gamma^TP^{-1}x_{c_0}. \tag{62}$$

Then the first order correction involves the integration of a $2n$-vector $[(\hat{x}_1^*)^T, (x_0^*)^T]$. It would appear that the dimension should grow on each iteration since the exact filter is infinite dimensional. However, the estimate itself using only zeroth-oder and the first order terms is $\hat{x}^* \cong \hat{x}_0^* + \epsilon\hat{x}_1^*$.

IV.2 Higher Order Correction Terms

Higher order correction terms for the approximate estimation scheme are derived similarly to the first order case. Thus most of the derivation details will not be shown. Only the important results are given. For $n \geq 2$, the correction terms for the approximate game-theoretic filter are derived as

$$
\begin{aligned}
\dot{\hat{x}}_n^* &= A\hat{x}_n^* + g_{n-1}^m - PH^TV^{-1}(h_{n-1}^m + H\hat{x}_n^*) - P(h_{x_{n-1}}^m)^TV^{-1}(-z + Hx_0^m) \\
&\quad - \sum_{i=0}^{n-2} P(h_{x_i}^m)^TV^{-1}(h_{n-2-i}^m + Hx_{n-1-i}^m) + \sum_{i=0}^{n-1} P(g_{x_i}^m)^T\lambda_{n-1-i}^m, \\
&\quad \hat{x}_n^*(0) = 0
\end{aligned} \tag{63}
$$

where P is same as in (38) and $x_n^* = x_{c_n} + P\lambda_n^*$ has been used to derive (63). In deriving (63), the expansions of J^1 as in (23) up to the order of magnitude ϵ^{2n} (not ϵ^n) are necessary for the minimization problem

$$
min_{\hat{x}_n} J^1 = min_{\hat{x}_n}(J_0 + \epsilon J_1 + \cdots + \epsilon^{2n} J_{2n}) \tag{64}
$$

subject to its related constraints. In (64), for $m = 1, ..., 2n$

$$
\begin{aligned}
J_m &= \theta \int_0^t \frac{\theta^{-1}}{2}(\sum_{i=0}^{m} r_i^T Q r_{m-i}) - (-z + Hx_{c_0}^*)^T V^{-1}(Hx_{c_m} + h_{m-1}^*) \\
&\quad -\frac{1}{2}[\sum_{i=0}^{m-2}(Hx_{c_{i+1}} + h_i^m)^T V^{-1}(Hx_{c_{m-1-i}} + h_{m-2-i}^m)] \\
&\quad -\sum_{i=0}^{m-1}(\lambda_{m-1-i}^*)^T P \Phi_i(x_0^*, ..., x_{m-1}^*) d\tau
\end{aligned} \tag{65}
$$

where

$$
\begin{aligned}
r_i &= x_{c_i} - \hat{x}_i, \ \ i = 0, 1, ..., 2n, & (66) \\
\Phi_i &= -(h_{x_i}^*)^T V^{-1}(-z + Hx_0^*) - \sum_{j=0}^{i-1}(h_{x_{i-1-j}})^T V^{-1}(Hx_{j+1}^* + h_j^*) \\
&\quad + \sum_{k=0}^{i}(g_{x_{i-k}^*})^T \lambda_k^*, \ \ i = 0, 1, ..., 2n - 1, & (67) \\
g_{x_i}^* &= h_{x_i}^* = 0, \ \ i < 0. & (68)
\end{aligned}
$$

V. Disturbance Attenuation and the Approximate Minimax Estimator

The approximate game-theoretic estimator using up to infinite order correction terms as derived in sections III and IV is first developed as a minimax estimator asympotically. Then, the disturbance attenuation problem as defined in section II is shown to be solved by this infinite order approximate game-theoretic filter. Based on these, a N-th order approximate game-theoretic estimator is then proved to satisfy a disturbance attenuation inequality with a higher disturbance attenuation threshold. To begin, J^* as in (23) is rewritten as

$$\begin{aligned} J^* &\equiv J_0(\hat{x}_0, x_0^*(0), w_0^*) + \epsilon J_1(\hat{x}_1, \hat{x}_0, x_1^*(0), x_0^*(0), w_1^*, w_0^*) \\ &\quad +\epsilon^2 J_2(\hat{x}_2, \hat{x}_1, \hat{x}_0, x_2^*(0), x_1^*(0), x_0^*(0), w_2^*, w_1^*, w_0^*) + \cdots \\ &= \theta \int_0^t \frac{\theta^{-1}}{2} \|(x_{c0} - \hat{x}_0) + \epsilon(x_{c1} - \hat{x}_1) + \epsilon^2(x_{c2} - \hat{x}_2) + \cdots \|_Q^2 \\ &\quad -\frac{1}{2}\|(-z + Hx_{c0}) + \epsilon(h_0^* + Hx_{c1}) + \epsilon^2(h_1^* + Hx_{c2}) + \cdots \|_{V^{-1}}^2 \\ &\quad -\epsilon\|(x_0^* - x_{c0}) + \epsilon(x_1^* - x_{c1}) + \ldots\|^2_{(g^*_{x_0}(\hat{x}_0)+\epsilon g^*_{x_1}(\hat{x}_1)+\ldots)^T P^{-1}} \\ &\quad +\epsilon[(x_0^* - x_{c0}) + \epsilon(x_1^* - x_{c1}) + \ldots]^T (h^*_{x_0}(\hat{x}_0) + \epsilon h^*_{x_1}(\hat{x}_1) + \ldots)^T V^{-1} \\ &\quad [(-z + Hx_0^*) + \epsilon(h_0^* + Hx_1^*) + \ldots] d\tau. \end{aligned} \tag{69}$$

(69) is obtained by first adding up the component equations as derived in (30), (40), etc. and then using the completing the square technique. Note that g^* and h^* are tacitly assumed to be continuous and differentiable. Thus, the cost using the minimax strategies derived in sections III and IV is

$$\begin{aligned} J^M &= J_0(\hat{x}_0^*, x_0^m(0), w_0^m) + \epsilon J_1(\hat{x}_1^*, \hat{x}_0^*, x_1^m(0), x_0^m(0), w_1^m, w_0^m) \\ &\quad +\epsilon^2 J_2(\hat{x}_2^*, \hat{x}_1^*, \hat{x}_0^*, x_2^m(0), x_1^m(0), x_0^m(0), w_2^m, w_1^m, w_0^m) + \cdots \\ &= \theta \int_0^t -\frac{1}{2}\|(-z + H\hat{x}_0^*) + \epsilon(h_0^m + H\hat{x}_1^*) + \epsilon^2(h_1^m + H\hat{x}_2^*) + \cdots \|_{V^{-1}}^2 \\ &\quad -\epsilon\|(x_0^m - \hat{x}_0^*) + \epsilon(x_1^m - \hat{x}_1^*) + \cdots \|^2_{(g^m_{x_0}(\hat{x}_0^*)+\epsilon g^m_{x_1}(\hat{x}_1^*)+\cdots)^T P^{-1}} \\ &\quad +\epsilon[(x_0^m - \hat{x}_0^*) + \epsilon(x_1^m - \hat{x}_1^*) + \ldots]^T (h^m_{x_0}(\hat{x}_0^*) + \epsilon h^m_{x_1}(\hat{x}_1^*) + \cdots)^T V^{-1} \\ &\quad [(-z + Hx_0^m) + \epsilon(h_0^m(\hat{x}_0^*) + Hx_1^m) + \cdots] d\tau \end{aligned} \tag{70}$$

After some algebra, (70) is recast as a perfect square expression as

$$\begin{aligned} J^M &= \theta \int_0^t -\frac{1}{2}\|(-z + H\hat{x}_0^*) + \epsilon(h_0^m + H\hat{x}_1^*) + \epsilon^2(h_1^m + H\hat{x}_2^*) + \cdots \|_{V^{-1}}^2 \\ &\quad -\epsilon\|(x_0^m - \hat{x}_0^*) + \epsilon(x_1^m - \hat{x}_1^*) + \cdots \|_G^2 \\ &\quad +\frac{\epsilon^2}{2}\|(h^m_{x_0} + \epsilon h^m_{x_1} + \cdots)[(x_0^m - \hat{x}_0^*) + \epsilon(x_1^m - \hat{x}_1^*) + \cdots]\|_{V^{-1}}^2 d\tau \end{aligned} \tag{71}$$

where

$$G \equiv (g^m_{x_0}(\hat{x}_0^*) + \epsilon g^m_{x_1}(\hat{x}_1^*) + \cdots)^T P^{-1} - (h^m_{x_0}(\hat{x}_0^*) + \epsilon h^m_{x_1}(\hat{x}_1^*) + \cdots)^T V^{-1} H. \tag{72}$$

Assumptions:

(1). $g(x)$ and $h(x)$ continuous and infinitely differentiable
(2). $h(0) = \frac{\partial h(x)}{\partial x}|_{x=0} = \cdots = 0$
(3). The matrix functions $A(\cdot), \Gamma(\cdot)$, and $H(\cdot)$ are continuous.

Theorem V.1:

For the dynamical system as given in (1) and (2), if Assumptions 1 and 2 are satisfied, then the infinite-order game-theoretic filter derived in previous sections solve the disturbance attenuation problem.

Proof of Theorem V.1:

For $\epsilon = 0$, the dynamical system (1) and (2) reduces to

$$\dot{x} = Ax + \Gamma w, \quad z = Hx + v \tag{73}$$

Therefore, the disturbance attenuation problem as defined in (4) degenerates to its corresponding linear problem [6, 9]. It is well known that the minimax strategy, obtained from an approach as used here for a linear problem, is unique when the initial state estimate vector is chosen as zero. Furthermore, from the estimation theory for a nonlinear dynamical system obtained from a small perturbation of a linear dynamical system as given in (73), there exists a sufficiently small perturbation such that the minimax estimation strategy is also unique. Now, since $(w_i, \hat{x}_i, x_i, z) = 0$, for $i = 0, 1, 2, \cdots$, satisfy the first order necessary condition as derived in section III and IV, thus zero is indeed the minimax trajectory produced by the unique minimax strategy. From (71), if Assumptions 1 and 2 are satisfied, then

$$J^M = 0.$$

Consequently, the strategies $\hat{x}_i^*, x_i^m(0)$, and w_i^m derived in sections III and IV are indeed the minimax estimator asymptotically. Naturally, when the adversaries do not play their minimax strategies,

$$\begin{aligned} J_* \equiv\ & J_0(\hat{x}_0^*, x_0(0), w_0) + \epsilon J_1(\hat{x}_1^*, \hat{x}_0^*, x_1(0), x_0(0), w_1, w_0) \\ & + \epsilon^2 J_2(\hat{x}_2^*, \hat{x}_1^*, \hat{x}_0^*, x_2(0), x_1(0), x_0(0), w_2, w_1, w_0) + \cdots \leq 0. \end{aligned} \tag{74}$$

Alternatively, (74) is recast as

$$D_{af}(\hat{x}^*, w, x(0)) \leq \theta, \tag{75}$$

for all $w \epsilon L_2[0, t]$ and $x(0) \epsilon R^n$ s.t. $(w(\tau), v(\tau)) \neq 0$ for all $\tau \epsilon [0, t]$ and $x(0) \neq \hat{x}(0)$, where the disturbance attenuation function for the infinite order approximate game-theoretic estimator is defined as in section II.

$\diamond$

Denote

$$\hat{x}^* = {}^N\hat{x}^* + {}^R\hat{x}^* = \sum_{i=0}^{N} \hat{x}_i^* \epsilon^i + \sum_{i=N+1}^{\infty} \hat{x}_i^* \epsilon^i, \tag{76}$$

Now, $D_{af}(\hat{x}^*, w, x(0))$ is decomposed as

$$D_{af}(\hat{x}^*, w, x(0)) = \frac{a+b}{c+d} \leq \theta, \tag{77}$$

where

$$a \equiv \int_0^t \|x - {}^N\hat{x}^*\|_Q^2 d\tau \tag{78}$$

$$b \equiv \int_0^t -2(x - {}^N\hat{x}^*)^T Q {}^R\hat{x}^* + \|{}^R\hat{x}^*\|_Q^2 d\tau \tag{79}$$

$$c \equiv \|x(0) - {}^N\hat{x}^*(0)\|_{P_0^{-1}}^2 + \int_0^t \|w\|_{W^{-1}}^2 + \|z - Hx - \epsilon h(x)\|_{V^{-1}}^2 d\tau \tag{80}$$

$$d \equiv -2[x(0) - {}^N\hat{x}^*(0)]^T P_0^{-1} {}^R\hat{x}^*(0) + \|{}^R\hat{x}^*(0)\|_{P_0^{-1}}^2. \tag{81}$$

Corresponding to the disturbance attenuation problem defined in section II, a N-th order disturbance attenuation problem for the N-th order approximate estimator truncated from the infinite order minimax estimator is defined as to find ${}^N\hat{x}^*$ such that

$$D_{af}({}^N\hat{x}^*, w, x(0)) \leq \theta' > \theta > 0 \tag{82}$$

for all $w \epsilon L_2[0,t]$ and $x(0) \epsilon R^n$ such that $(w(\tau), v(\tau)) \neq 0$ for all $\tau \epsilon [0,t]$ and $x(0) \neq \hat{x}(0)$ and where

$$D_{af}({}^N\hat{x}, w, x(0)) \equiv \frac{\int_0^t \|x - {}^N\hat{x}\|_Q^2 d\tau}{\|x(0) - {}^N\hat{x}(0)\|_{P_0^{-1}}^2 + \int_0^t \|w\|_{W^{-1}}^2 + \|z - Hx - \epsilon h(x)\|_{V^{-1}}^2 d\tau}. \tag{83}$$

Not that ${}^N\hat{x}^*$ is the truncated state estimate vector as defined in (76). Note that, from (78) and (80),

$$D_{af}({}^N\hat{x}, w, x(0)) = \frac{a}{c}.$$

Theorem V.2:

For the dynamical system as in (1) and (2), if the disturbance attenuation problem (4) is solved by the infinite order minimax estimator, then, the N-th order disturbance attenuation problem as defined in (82) is solved by the N-th order estimator, ${}^{N}\hat{x}^{*}$, with a threshold which is at most twice of the original threshold.

Proof of Thm V.2:

From (77), using the fact that $c + d > 0$, thus,

$$\frac{a}{c} \leq \theta + \frac{-b + \theta d}{c} \leq \theta + \frac{|\theta d - b|}{c} \tag{84}$$

In (81), ${}^{R}\hat{x}^{*}(0) = 0$, therefore $d = 0$, (84) is further reduced to

$$\frac{a}{c} \leq \theta + \frac{|b|}{c} \leq \theta + \frac{a+b}{c+d} \leq 2\theta. \tag{85}$$

$\diamondsuit$

Remarks V.3:

Since N is arbitrary, (85) shows that the disturbance attenuation inequality is always bounded by 2θ no matter how many terms are used in the approximate game-theoretic filter.

$\diamondsuit$

Theorem V.4:

For the dynamical system as given in (1) and (2), if the disturbance attenuation problem is solved by the infinite order minimax estimator and assumption (3) is satisfied, then the N-th order disturbance attenuation problem (82) is sloved by the N-th order approximate estimator. Moreover, the upper bound of the threshold is proportional to $N + 1$ power of ϵ.

Proof of Theorem V.4:

Under assumption 3, one can show that there exists $0 < K_1 < \infty$ s.t.

$$|b| < K_1 \epsilon^{N+1}, \tag{86}$$

where b is defined as in (79). From (80), c is a zeroth-order term. From Theorem V.2, $\frac{|b|}{c} \leq \theta$. Therefore, there exists K_2 such that $\frac{K_1}{c} \leq K_2$. Thus

$$\frac{|b|}{c} < \frac{K_1 \epsilon^{N+1}}{c} \leq K_2 \epsilon^{N+1} \leq \theta. \tag{87}$$

◊

Remarks V.5:

As $N \to \infty$, $K_2\epsilon^{N+1} \to 0$, thus $\frac{|b|}{c} \to 0$. Therefore, as $N \to \infty$,

$$D_{af}({}^N\hat{x}^*, w, x(0)) = \frac{a}{c} \leq \theta + \frac{|b|}{c} \to \theta. \tag{88}$$

This recovers the infinite order threshold θ.

◊

VI. Extension to the Measurement Feedback Control Case

An extension of the approximate game-theoretic estimator as derived in sections III and IV is presented here. In [9], a linear game-theoretic controller is derived via solving a disturbance attenuation problem. To generalize, consider a small perturbation of the dynamical system in [9] as

$$\dot{x} = Ax + Bu + \Gamma w + \epsilon f(x) \tag{89}$$

$$z = Hx + \Gamma_1 v + \epsilon q(x) \tag{90}$$

First, a disturbance attenuation function is defined as

$$\bar{D}_{af}(u, v, w, x(0)) = \frac{\|x(T)\|^2_{Q_T} + \int_0^T \|x\|^2_Q + \|u\|^2_R d\tau}{\|x(0) - \hat{x}(0)\|^2_{P_0^{-1}} + \int_0^T \|w\|^2_{W^{-1}} + \|v\|^2_{V^{-1}} d\tau}. \tag{91}$$

Based on the definition of (91), the control disturbance attenuation problem is defined as to find an approximate optimal ocntroller u^* s.t.

$$\bar{D}_{af}(u^*, v, w, x(0)) \leq \bar{\theta}, \ \ \bar{\theta} > 0, \tag{92}$$

for all $w, v \epsilon L_2[0, T]$ and $x(0) \epsilon R^n$ such that $(w(\tau), v(\tau)) \neq 0$ for all $\tau \epsilon [0, T]$ and $x(0) \neq \hat{x}(0)$, simultaneously.

Similarly as in [9], to obatin a measurement feedback control based on causal mapping of the measurement sequence, consider the minimax optimization problem

$$min_u max_v max_{w,x(0)} \bar{J} \leq 0 \tag{93}$$

subject to (89) and (90) where

$$\begin{aligned} \bar{J} \ &= \ -\frac{\theta}{2}\|x(0) - \hat{x}(0)\|^2_{P_0^{-1}} + \frac{1}{2}\|x(T)\|^2_{Q_T} \\ &+\frac{1}{2}\int_0^T \|x\|^2_Q + \|u\|^2_R - \theta(\|w\|^2_{W^{-1}} + \|v\|^2_{V^{-1}}) d\tau. \end{aligned} \tag{94}$$

In (94), T is the terminal time which is different from the present time t in the estimation problem. Note also that the maximization of v is not trivial in this problem due to the unknown measurement sequence in the future.

First, begin with the maximization problem for fixed u and v

$$\bar{J}^* = max_{w,x(0)}\bar{J} \tag{95}$$

subject to (89) and (90). $\bar{J}$ is rewritten as

$$\begin{aligned} \bar{J} &= -\frac{\theta}{2}\|x(0)-\hat{x}(0)\|^2_{\bar{P}_0^{-1}} + \frac{1}{2}\|x(T)\|^2_{Q_T} + \\ & \frac{1}{2}\int_0^T \|x\|^2_Q + \|u\|^2_R - \theta\|w\|^2_{W^{-1}} - \theta\|z - Hx - \epsilon q\|^2_{\bar{V}^{-1}} d\tau \end{aligned} \tag{96}$$

where $\bar{V} = \Gamma_1 V \Gamma_1^T$ (Γ_1 nonsingular). Furthermore, adjoin (89) to $\bar{J}$ using a Lagrange multiplier ρ. The first order necessary condition for problem (95) follows easily as

$$\begin{aligned} x(0) &= \hat{x}(0) + \bar{P}_0\rho(0),\ \rho(T) = \theta^{-1}Q_T x(T),\ w = W\Gamma^T\rho, \\ \dot{\rho} &= -\epsilon f_x^T\rho - A^T\rho - (H+\epsilon q_x)^T\bar{V}^{-1}(z - Hx - \epsilon q_x) - \theta^{-1}Qx. \end{aligned} \tag{97}$$

Next, form a nonlinear two point boundary value problem from (97) and (89) as

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\rho} \end{bmatrix} &= \begin{bmatrix} A & \Gamma W \Gamma^T \\ (H+\epsilon q_x)^T\bar{V}^{-1}H - \theta^{-1}Q & -A^T - \epsilon(f_x)^T \end{bmatrix} \begin{bmatrix} x \\ \rho \end{bmatrix} \\ &+ \begin{bmatrix} \epsilon f + Bu \\ -(H+\epsilon q_x)^T\bar{V}^{-1}(z-\epsilon q) \end{bmatrix}, \\ & x(0) = \hat{x}(0) + \bar{P}_0\rho(0),\ \ \rho(T) = \theta^{-1}Q_T x(T). \end{aligned} \tag{98}$$

To obtain an approximate game-theoretic control scheme, expand x, ρ, and u as

$$x = \sum_{j=0}^{\infty} x_j\epsilon^j,\ \ \rho = \sum_{j=0}^{\infty}\rho_j\epsilon^j,\ \ u = \sum_{j=0}^{\infty} u_j\epsilon^j. \tag{99}$$

Also, from Taylor's series expansion, let

$$f = \sum_{j=0}^{\infty} f_j\epsilon^j,\ ,q = \sum_{j=0}^{\infty} q_j\epsilon^j,\ f_x = \sum_{j=0}^{\infty} f_{x_j}\epsilon^j,\ q_x = \sum_{j=0}^{\infty} q_{x_j}\epsilon^j. \tag{100}$$

After substitution of the expansion, we have a series of linear two point boundary problem as

$$
\begin{aligned}
\epsilon^0 : \begin{bmatrix} \dot{x}_0 \\ \dot{\rho}_0 \end{bmatrix} &= \begin{bmatrix} A & \Gamma W \Gamma^T \\ H^T \bar{V}^{-1} H - \theta^{-1} Q & -A^T \end{bmatrix} \begin{bmatrix} x_0 \\ \rho_0 \end{bmatrix} + \begin{bmatrix} B u_0 \\ -H^T \bar{V}^{-1} z \end{bmatrix} \\
x_0(0) &= \hat{x}(0) + \bar{P}_0 \rho_0(0), \;\; \rho_0(T) = \theta^{-1} Q_T x_0(T) \qquad (101) \\
\epsilon^1 : \begin{bmatrix} \dot{x}_1 \\ \dot{\rho}_1 \end{bmatrix} &= \begin{bmatrix} A & \Gamma W \Gamma^T \\ H^T \bar{V}^{-1} H - \theta^{-1} Q & -A^T \end{bmatrix} \begin{bmatrix} x_1 \\ \rho_1 \end{bmatrix} + \\
& \begin{bmatrix} f_0 + B u_1 \\ -f_{x_0}^T \rho_0 + H^T \bar{V}^{-1} q_0 - q_{x_0}^T \bar{V}^{-1} (z - H x_0) \end{bmatrix} \\
x_1(0) &= \bar{P}_0 \rho_1(0), \;\; \rho_1(T) = \theta^{-1} Q_T x_1(T) \qquad (102) \\
\epsilon^2 : \begin{bmatrix} \dot{x}_2 \\ \dot{\rho}_2 \end{bmatrix} &= \begin{bmatrix} A & \Gamma W \Gamma^T \\ H^T \bar{V}^{-1} H - \theta^{-1} Q & -A^T \end{bmatrix} \begin{bmatrix} x_2 \\ \rho_2 \end{bmatrix} + \\
& \begin{bmatrix} f_1 + B u_2 \\ q_{x_0}^T \bar{V}^{-1} (H x_1 + q_0) - q_{x_1}^T \bar{V}^{-1} (z - H x_0) - f_{x_0}^T \rho_1 - f_{x_1}^T \rho_0 + H^T \bar{V}^{-1} q_1 \end{bmatrix} \\
x_2(0) &= \bar{P}_0 \rho_2(0), \;\; \rho_2(T) = \theta^{-1} Q_T x_2(T) \qquad (103)
\end{aligned}
$$

Furthermore, let x_j^* and ρ_j^*, called the worst case jth order state and Lagrange multiplier, be the solution to the corresponding problem in (103). Assume

$$
x_j^* = \hat{x}_j + \bar{P} \rho_j^*, \;\; j = 0, 1, 2, \ldots \qquad (104)
$$

Based on x_j^* and ρ_j^*, $\bar{J}^*$ is expanded as

$$
\begin{aligned}
\bar{J}^* &= -\frac{\theta}{2} \| \sum_{j=0}^{\infty} \rho_j^*(0) \epsilon^j \|_{\bar{P}_0}^2 + \frac{1}{2} \| \sum_{j=0}^{\infty} x_j^*(T) \epsilon^j \|_{Q_T}^2 + \frac{1}{2} \int_0^T \| \sum_{j=0}^{\infty} x_j^* \epsilon^j \|_Q^2 \\
& + \| \sum_{j=0}^{\infty} u_j \epsilon^j \|_R^2 - \theta \| \sum_{j=0}^{\infty} \rho_j^* \epsilon^j \|_{\Gamma W \Gamma^T}^2 - \theta \| z - H (\sum_{j=0}^{\infty} x_j^* \epsilon^j) - \epsilon (\sum_{j=0}^{\infty} q_j^* \epsilon^j) \|_{\bar{V}^{-1}}^2 \qquad (105)
\end{aligned}
$$

where the expansion series

$$
x^* = \sum_{j=0}^{\infty} x_j^* \epsilon^j, \;\; \rho^* = \sum_{j=0}^{\infty} \rho_j^* \epsilon^j, \;\; q(x^*) = \sum_{j=0}^{\infty} q_j^* \epsilon^j \qquad (106)
$$

are used. Denote $\bar{J}^* = \sum_{j=0}^{\infty} \bar{J}_k^* \epsilon^k$. Thus

$$
\begin{aligned}
\bar{J}_0^* &= -\frac{\theta}{2} \| \rho_0^*(0) \|_{\bar{P}_0}^2 + \frac{1}{2} \| x_0^*(T) \|_{Q_T}^2 + \\
& \frac{1}{2} \int_0^T \| x_0^* \|_Q^2 + \| u_0 \|_R^2 - \theta \| \rho_0^* \|_{\Gamma W \Gamma^T}^2 - \theta \| z - H x_0^* \|_{\bar{V}^{-1}}^2 d\tau,
\end{aligned}
$$

$$
\begin{aligned}
\bar{J}_1^* &= -\theta[\rho_0^*(0)]^T \bar{P}_0 \rho_1^*(0) + [x_0^*(0)]^T Q_T x_1^*(T) + \\
&\int_0^T (x_0^*)^T Q x_1^* + u_0^T R u_1 - \theta(\rho_0^*)^T \Gamma W \Gamma^T \rho_1^* - \theta(z - Hx_0^*)^T \bar{V}^{-1}(-Hx_1^* - q_0^*) d\tau, \\
\bar{J}_2^* &= -\theta[\rho_0^*(0)]^T \bar{P}_0 \rho_2^*(0) - \frac{\theta}{2}\|\rho_1^*(0)\|_{\bar{P}_0}^2 + [x_0^*(T)]^T Q_T x_2^*(T) + \frac{1}{2}\|x_1^*\|_{Q_T}^2 \\
&+ \int_0^T (x_0^*)^T Q x_2^* + \frac{1}{2}\|x_1^*\|_Q^2 + u_0^T R u_2 + \frac{1}{2}\|u_1\|_R^2 - \theta(\rho_0^*)^T \Gamma W \Gamma^T \rho_2^* - \\
&\theta\|\rho_1^*\|_{\Gamma W \Gamma^T}^2 - \theta(z - Hx_0^*)^T \bar{V}^{-1}(-Hx_2^* - q_1^*) - \frac{\theta}{2}\|Hx_1^* + q_0^*\|_{\bar{V}^{-1}}^2 d\tau. \qquad (107)
\end{aligned}
$$

VI.1 Zeroth-order Solution - the Linear Game-theoretic Controller

Consider the ϵ^0 term of (103). As in the estimation problem previlously, we obtain from sweep method using the assumption (104)

$$
\begin{aligned}
\dot{\hat{x}}_0 &= A\hat{x}_0 + Bu_0 + \bar{P}H^T\bar{V}^{-1}(z - H\hat{x}_0) + \theta^{-1}\bar{P}Q\hat{x}_0, \ \ \hat{x}_0(0) = \hat{x}(0) \qquad (108) \\
\dot{\bar{P}} &= A\bar{P} + \bar{P}A^T + \Gamma W \Gamma^T - \bar{P}(H^T\bar{V}^{-1}H - \theta^{-1}Q)\bar{P}. \qquad (109)
\end{aligned}
$$

To determine the zerith-order term of the approximate game-theoretic control, add the zero identity

$$
0 = \frac{\theta}{2}[\rho_0^*(0)]^T \bar{P}_0 \rho_0^*(0) - \frac{\theta}{2}[\rho_0^*(T)]^T \bar{P}(T)\rho_0^*(T) + \frac{\theta}{2}\int_0^T \frac{d}{d\tau}([\rho_0^*]^T \bar{P}\rho_0^*)d\tau) \qquad (110)
$$

to $\bar{J}_0^*$ as in (107). Thus

$$
\bar{J}_0^* = \frac{1}{2}\|\hat{x}_0(T)\|_{S_T}^2 + \frac{1}{2}\int_0^T \|\hat{x}_0\|_Q^2 + \|u_0\|_R^2 - \theta\|z - H\hat{x}_0\|_{\bar{V}^{-1}}^2 d\tau \qquad (111)
$$

where S_T can be evaluated from

$$
\begin{aligned}
\frac{1}{2}\|\hat{x}_0(T)\|_{S_T}^2 &= -\frac{\theta}{2}[\rho_0^*(T)]^T \bar{P}(T)\rho_0^*(T) + \frac{1}{2}\|x_0^*(T)\|_{Q_T}^2 \qquad (112) \\
S_T &\equiv Q_T[I - \theta^{-1}\bar{P}(T)Q_T]^{-1}. \qquad (113)
\end{aligned}
$$

Now, consider the minimax game problem

$$
min_{u_0} max_{\hat{v}_0} \bar{J}_0^* \qquad (114)
$$

subject to (108) where $\hat{v}_0 \equiv z - H\hat{x}_0$. For conveniences, (108) is rewritten as

$$
\dot{\hat{x}}_0 = \bar{A}\hat{x}_0 + Bu_0 + \bar{\Gamma}\hat{v}_0 \qquad (115)
$$

where

$$\bar{A} \equiv A + \theta^{-1}\bar{P}Q, \;\; \bar{\Gamma} \equiv \bar{P}H^T\bar{V}^{-1}. \tag{116}$$

Obviously, from [8], the zeroth-order correction of the approximate game-theoretic measurement feedback controller is

$$\begin{aligned} u_0^* &= -R^{-1}B^TS\hat{x}_0 & (117)\\ \hat{v}_0^* &= \theta^{-1}\bar{V}\bar{\Gamma}^TS\hat{x}_0 = \theta^{-1}H\bar{P}S\hat{x}_0 & (118)\end{aligned}$$

where S is determined from

$$\begin{aligned} -\dot{S} &= S\bar{A} + \bar{A}^TS - S(BR^{-1}B^T - \theta^{-1}\bar{\Gamma}\bar{V}\bar{\Gamma}^T)S + Q \\ &= S(A+\theta^{-1}\bar{P}Q) + (A+\theta^{-1}\bar{P}Q)^TS - S(BR^{-1}B^T - \theta^{-1}\bar{P}H^T\bar{V}^{-1}H\bar{P})S + Q \\ S(T) &= S_T & (119)\end{aligned}$$

VI.2 First Order Correction Term of the Game-theoretic Controller

Consider the ϵ^1 term of (103). As previously in the estimation problem, using sweep method, the first order correction term of the approximate game-theoretic controller is derived via the minimax game problem of $\bar{J}_2^*$ with respect to the first order correction terms of the controller and residual. Note that as previously in the estimation problem the minimax game problem of $\bar{J}_1^*$ with respect to the first order terms of the controller and residual is not significant for the similar reason. First, the first order estimation correction term of the approximate measurement feedback controller is derived as

$$\begin{aligned} \dot{\hat{x}}_1 &= A\hat{x}_1 + Bu_1 + f_0^m + \bar{P}(q_{x_0}^m)^T\bar{V}^{-1}(z - Hx_0^m) - \bar{P}H^T\bar{V}^{-1}(H\hat{x}_1 + q_0^m) + \theta^{-1}\bar{P}Q\hat{x}_1 \\ &\quad + \bar{P}(f_{x_0}^m)^T\rho_0^m, \;\; \hat{x}_1(0) = 0, & (120)\\ \dot{\bar{P}} &= A\bar{P} + \bar{P}A^T + \Gamma W\Gamma^T - \bar{P}(H^T\bar{V}^{-1}H - \theta^{-1}Q)\bar{P}, \;\; \bar{P}(0) = \bar{P}_0. & (121)\end{aligned}$$

To proceed, after substituting u_0^* and $\hat{v}_0^*$ into $\bar{J}_2^*$ as in (107) and using an appropriately chosen zero identity as before, $\bar{J}_2^*$ is converted to

$$\bar{J}_2^*(u_0^*, \hat{v}_0^*) \;=\; (\hat{x}_0^*)^T(T)S_T\hat{x}_2(T) + \frac{1}{2}\|\hat{x}_1(T)\|_{S_T}^2 + \theta\int_0^T \bar{I}_2 d\tau \tag{122}$$

where

$$\bar{I}_2 = \theta^{-1}(\hat{x}_0^*)^T Q\hat{x}_2 + \theta^{-1}(u_0^*)^T Ru_2 + (z - H\hat{x}_0^*)^T\bar{V}^{-1}(H\hat{x}_2 + q_1^*) + (\rho_0^m)^T\bar{P}[(q_{x_0}^m)^T\bar{V}^{-1}$$
$$(q_0^m + Hx_1^*) - (q_{x_1}^*)^T\bar{V}^{-1}(z - Hx_0^m) - (f_{x_0}^m)^T\bar{P}^{-1}(x_1^* - \hat{x}_1) - (f_{x_1}^*)^T\rho_0^m]$$
$$\frac{\theta^{-1}}{2}\|\hat{x}_1\|_Q^2 + \frac{\theta^{-1}}{2}\|u_1\|_R^2 - \frac{1}{2}\|H\hat{x}_1 + q_0^m\|_{\bar{V}^{-1}}^2$$
$$-(x_1^* - \hat{x}_1)^T[(q_{x_0}^m)^T\bar{V}^{-1}(z - Hx_0^m) + (f_{x_0}^m)^T\rho_0^m] \tag{123}$$

To obtain the first order correction term of the approximate control, consider the minimization problem

$$min_{u_1}\bar{J}_2^*(\hat{x}_1, \hat{x}_2, u_1, u_2, x_1^*) \tag{124}$$

subject to

$$\dot{\hat{x}}_1 = \tilde{A}\hat{x}_1 + Bu_1 + \tilde{v}_{01}, \ \hat{x}_1(0) = 0, \tag{125}$$
$$\dot{\hat{x}}_2 = \tilde{A}\hat{x}_2 + Bu_2 + \tilde{v}_{02}, \ \hat{x}_2(0) = 0, \tag{126}$$
$$\dot{x}_1^* = Ax_1^* + \Gamma W\Gamma^T\bar{P}^{-1}(x_1^* - \hat{x}_1) + Bu_1 + f_0^m, \tag{127}$$
$$x_1^*(T) = [I - \theta^{-1}\bar{P}(T)Q_T]^{-1}\hat{x}_1(T) \tag{128}$$

where

$$\tilde{A} = A + \theta^{-1}\bar{P}Q - \bar{P}H^T\bar{V}^{-1}H, \tag{129}$$
$$\tilde{v}_{01} = -\bar{P}H^T\bar{V}^{-1}q_0^m + \bar{P}(q_{x_0}^m)^T\bar{V}^{-1}(z - Hx_0^m) + f_0^m + \bar{P}(f_{x_0}^m)^T\rho_0^m \tag{130}$$
$$\tilde{v}_{02} = \bar{P}[-H^T\bar{V}^{-1}q_1^* + (q_{x_1}^*)^T\bar{V}^{-1}(z - Hx_0^m) - (q_{x_0}^m)^T\bar{V}^{-1}(Hx_1^* + q_0^m)$$
$$+(f_{x_1}^*)^T\rho_0^m + (f_{x_0}^m)^T\rho_1^*] + f_1^* \tag{131}$$

Next, using Lagrange Multiplier Techniques, the 1st order optimality condition for the problem (124) is derived as

$$\dot{\bar{\beta}} = -A^T\bar{\beta} - \bar{P}^{-1}\Gamma W\Gamma^T\bar{\beta} - [(z - Hx_0^m)^T\bar{V}^{-1}\frac{\partial q_{x_1}^*}{\partial x_1^*}\bar{P} - H^T\bar{V}^{-1}q_{x_0}^m\bar{P}$$
$$+(\rho_0^m)^T\frac{\partial f_{x_1}^*}{\partial x_1^*}\bar{P} + \bar{P}^{-1}f_{x_0}^m\bar{P}](\bar{\sigma} - \rho_0^m) + (\frac{\partial q_1^*}{\partial x_1^*})^T[\bar{\Gamma}^T\bar{\sigma} - \bar{V}^{-1}\hat{v}_0]$$
$$+(q_{x_0}^m)^T\bar{V}^{-1}(z - Hx_0^m) - (\frac{\partial f_1^*}{\partial x_1^*})^T\bar{\sigma} + (f_{x_0}^m)^T\rho_0^m, \ \bar{\beta}(0) = \bar{\beta}(T) = 0, \tag{132}$$
$$\dot{\bar{\gamma}} = -\tilde{A}^T\bar{\gamma} - \theta^{-1}Q\hat{x}_1 - (f_{x_0}^m)^T\rho_0^m - (q_{x_0}^m)^T\bar{V}^{-1}(z - Hx_0^m) + \bar{P}^{-1}\Gamma W\Gamma^T\bar{\beta}$$
$$+\bar{P}^{-1}f_{x_0}^m\bar{P}(\bar{\sigma} - \rho_0^m) + H^T\bar{V}^{-1}(H\hat{x}_1 + q_0^m), \ \bar{\gamma}(T) = \theta^{-1}S_T\hat{x}_1(T), \tag{133}$$
$$\dot{\bar{\sigma}} = -\tilde{A}^T\bar{\sigma} - \theta^{-1}Q\hat{x}_0^* - H^T\bar{V}^{-1}\hat{v}_0^*, \ \bar{\sigma}(T) = \theta^{-1}S_T\hat{x}_0^*(T), \tag{134}$$
$$u_1^* = -\theta R^{-1}B^T(\bar{\beta} + \bar{\gamma}), \tag{135}$$

From (101),

$$\dot{\rho}_0^m = -\tilde{A}^T \rho_0^m - \theta^{-1} Q \hat{x}_0^* - H^T \bar{V}^{-1} \hat{v}_0^*, \;\; \rho_0^m(T) = \theta^{-1} S_T \hat{x}_0^*(T). \qquad (136)$$

Comparison of (134) and (136) reveals that $\rho_0^m(\tau) = \bar{\sigma}(\tau)$, for all τ. In addition, note that

$$[\frac{\partial f_1^*(x_0^m)}{\partial x_1^*}]^T = (f_{x_0}^m)^T; \; [\frac{\partial q_1^*(x_0^m)}{\partial x_1^*}]^T = (q_{x_0}^m)^T; \; \bar{\Gamma}^T \bar{\sigma} - \bar{V}^{-1} \hat{v}_0^* = -\bar{V}^{-1}(z - H x_0^m). \quad (137)$$

Substitution of (137) into (132) yields

$$\dot{\bar{\beta}} = -A^T \bar{\beta} - \bar{P}^{-1} \Gamma W \Gamma^T \bar{\beta}, \;\; \bar{\beta}(0) = \bar{\beta}(T) = 0. \qquad (138)$$

Thus, $\bar{\beta}(\tau) = 0$, for all τ. Clearly, (133) is reduced to

$$\dot{\bar{\gamma}} = -\tilde{A}^T \bar{\gamma} - \theta^{-1} Q \hat{x}_1 - (f_{x_0}^m)^T \rho_0^m - (q_{x_0}^m) T \bar{V}^{-1}(z - H x_0^m) + H^T \bar{V}^{-1}(H \hat{x}_1 + q_0^m). \quad (139)$$

From (102), after some algebra,

$$\begin{aligned} \dot{\rho}_1^m \;\; &= \;\; -\tilde{A}^T \rho_1^m + (H^T \bar{V}^{-1} H - \theta^{-1} Q) \hat{x}_1^* + H^T \bar{V}^{-1} q_0^m - (q_{x_0}^m)^T \bar{V}^{-1}(z - H x_0^m) \\ & \quad -(f_{x_0}^m)^T \rho_0^m, \;\; \rho_1^m(T) = \theta^{-1} S_T \hat{x}_1^*(T), \qquad (140) \end{aligned}$$

Comparison of (139) and (140) reveals that for all τ,

$$\bar{\gamma}^*(\tau) = \rho_1^m(\tau). \qquad (141)$$

Substitution of $\bar{\beta}(\tau) = 0$ and (141) into (135) yields

$$u_1^* = -\theta R^{-1} B^T \rho_1^m = -\theta R^{-1} B^T \bar{P}^{-1}(x_1^m - \hat{x}_1^*), \qquad (142)$$

where $\bar{P}$ is calculated from (121). Unfortunately, (142) is not implementable since both x_1^m and $\hat{x}_1^*$ are required to calculate u_1^*.

Alternatively, form a two-point boundary value problem from (125) and (139) as

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\bar{\gamma}} \end{bmatrix} &= \begin{bmatrix} \tilde{A} & -\theta B R^{-1} B^T \\ H^T \bar{V}^{-1} H - \theta^{-1} Q & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \bar{\gamma} \end{bmatrix} + \\ & \quad \begin{bmatrix} \tilde{v}_{01} \\ H^T \bar{V}^{-1} q_0^m - (f_{x_0}^m)^T \rho_0^m - (q_{x_0}^m)^T \bar{V}^{-1}(z - H x_0^m) \end{bmatrix} \qquad (143) \\ \hat{x}_1(0) &= 0, \;\; \bar{\gamma}(T) = \theta^{-1} S_T \hat{x}_1(T). \qquad (144) \end{aligned}$$

As previously, let the solution of (143) and (144) be $\hat{x}_1^*$ and $\bar{\gamma}^*$. Using sweep method, assume

$$\hat{x}_1^* = \tilde{x}_1 + \theta \bar{S}^{-1} \bar{\gamma}^*. \tag{145}$$

The choices of

$$\begin{aligned} -\dot{\bar{S}} &= \bar{S}\tilde{A} + \tilde{A}^T \bar{S} - \bar{S} B R^{-1} B^T \bar{S} - \theta (H^T \bar{V}^{-1} H - \theta^{-1} Q), \ \bar{S}(T) = S_T & (146) \\ \dot{\tilde{x}}_1 &= \check{A} \tilde{x}_1 + \check{v}_{01}, \ \tilde{x}_1(T) = 0 & (147) \end{aligned}$$

render (145) as an identity, where

$$\check{A} = A + \theta^{-1} M Q - M H^T \bar{V}^{-1} H; \ M = \bar{P} + \theta \bar{S}^{-1}; \tag{148}$$

$$\check{v}_{01} \equiv -M H^T \bar{V}^{-1} q_0^m + M (q_{x_0}^m)^T \bar{V}^{-1} (z - H x_0^m) + M (f_{x_0}^m)^T \rho_0^m + f_0^m. \tag{149}$$

Finally, the first order correction terms of the approximate game-theoretic controller and the residual are derived as

$$u_1^* = -R^{-1} B^T \bar{S} (\hat{x}_1^* - \tilde{x}_1) \tag{150}$$

Note that in (147) boundary condition is at the terminal time. Therefore, in order to implement u_1^*, for the time interval from the present time to future we need to rewrite $z - H x_0^m$ as

$$z - H x_0^m = \hat{v}_0^* + H(\dot{\hat{x}}_0^* - x_0^m), \tag{151}$$

due to the fact that the measurement vector in the future is not available to the designer.

VI.3 Higher Order Correction Terms of the Approximate Game-theoretic Controller

Similar to the first order case, higher order correction terms of the approximate game-theoretic control and residual is derived as, for $n \leq 2$, (using $\hat{x}_n^* = \tilde{x}_n + \theta \bar{S}^{-1} \bar{\gamma}_n^*$)

$$u_n^* = -R^{-1} B^T \bar{S} (\hat{x}_n^* - \tilde{x}_n); \tag{152}$$

where

$$\begin{aligned} \dot{\hat{x}}_n &= \tilde{A} \hat{x}_n + B u_n + \tilde{v}_{0n}, \ \hat{x}_n(0) = 0, & (153) \\ \dot{\tilde{x}}_n &= \check{A} \tilde{x}_n + \check{v}_{0n}, \ \tilde{x}_n(0) = 0, & (154) \\ & & (155) \end{aligned}$$

$$\tilde{v}_{0n} = -\bar{P}[\sum_{i=0}^{n-1}(q^*_{x_i})^T\bar{V}^{-1}(Hx^*_{n-1-i} + q^*_{n-2-i}) - (f^*_{x_i})^T\rho^*_{n-1-i}] + f^*_{n-1}, \quad (156)$$

$$\check{v}_{0n} = -M[\sum_{i=0}^{n-1}(q^*_{x_i})^T\bar{V}^{-1}(Hx^*_{n-1-i} + q^*_{n-2-i}) - (f^*_{x_i})^T\rho^*_{n-1-i}] + f^*_{n-1}, \quad (157)$$

$$q^*_{-1} \equiv -z. \quad (158)$$

Furthermore, $\bar{P}$ and $\bar{S}$ are evaluated from (121) and (146), respectively.

VI.4 Disturbance Attenuation and the Approximate Controller

The disturbance attenuation property for the approximate game-theoretic controller derived in subsections VI.2 and VI.3 is established. Two disturbance attenuation properties are presented here. First, the disturbance attenuation property for the infinite order controller is proved. Based on this, the disturbance attenuation property for the finite order approximate controller is shown. For the later case, the threshold is first derived as twice of the original threshold. Then, as the number of correction terms increased, this threshold is proved to shrink and converge to the original one when infinite order controller is used.

The disturbance attenuation property of the approximate controller using infinite correction terms can be proved similarly as the estimation problem in section V.

To start, $\bar{J}^*$ as in (105) is rewritten as

$$\begin{aligned}\bar{J}^* = & \frac{1}{2}\|\hat{x}_0(T) + \epsilon\hat{x}_1(T) + \cdots\|^2_{S_T} + \theta\int_0^T \frac{\theta^{-1}}{2}\|\hat{x}_0 + \epsilon\hat{x}_1 + \cdots\|^2_Q + \frac{\theta^{-1}}{2}\|u_0 + \epsilon u_1 + \cdots\|^2_R \\ & -frac12\|(z - H\hat{x}_0) + \epsilon(-q^*_0 - H\hat{x}_1) + \epsilon^2(-q^*_1 - H\hat{x}_2) + \cdots\|^2_{\bar{V}^{-1}} \\ & -\epsilon\|(x^*_0 - \hat{x}_0) + \epsilon(x^*_1 - \hat{x}_1) + \cdots\|_{(f^*_{x_0} + \epsilon f^*_{x_1} + \cdots)^T\bar{P}^{-1}} \\ & +\epsilon[(x^*_0 - \hat{x}_0) + \epsilon(x^*_1 - \hat{x}_1 + \cdots]^T(q^*_{x_0} + \epsilon q^*_{x_1} + \cdots)^T\bar{V}^{-1}[(-z + Hx^*_0) + \epsilon(q^*_0 + Hx^*_1) + \cdots]d\tau.\end{aligned}$$

(159)

Suppose u_i, for $i = 0, 1, \cdots$, play their minimax strategies, then

$$\begin{aligned}\bar{J}^M = & \frac{1}{2}\|\hat{x}^*_0(T) + \epsilon\hat{x}^*_1(T) + \cdots\|^2_{S_T} + \theta\int_0^T \frac{\theta^{-1}}{2}\|\hat{x}^*_0 + \epsilon\hat{x}^*_1 + \cdots\|^2_Q + \frac{\theta^{-1}}{2}\|u^*_0 + \epsilon u^*_1 + \cdots\|^2_R \\ & -frac12\|(z - H\hat{x}^*_0) + \epsilon(-q^m_0 - H\hat{x}^*_1) + \epsilon^2(-q^m_1 - H\hat{x}^*_2) + \cdots\|^2_{\bar{V}^{-1}} \\ & -\epsilon\|(x^m_0 - \hat{x}^*_0) + \epsilon(x^m_1 - \hat{x}^*_1) + \cdots\|_{(f^m_{x_0} + \epsilon f^m_{x_1} + \cdots)^T\bar{P}^{-1}} \\ & +\epsilon[(x^m_0 - \hat{x}^*_0) + \epsilon(x^m_1 - \hat{x}^*_1) + \cdots]^T(q^m_{x_0} + \epsilon q^m_{x_1} + \cdots)^T\bar{V}^{-1}[(-z + Hx^m_0) + \epsilon(q^m_0 + Hx^m_1) + \cdots\end{aligned}$$

(]

Assumptions:

(4). $f(x)$ and $q(x)$ continuous and infinitely differentiable
(5). $q(0) = q_x(0) = \cdots = 0$

Theorem VI.1:

For the Dynamical system as given in (89) and (90), if Assumptions (4) and (5) are satisfied, then the infinite-order game-theoretic controller derived in sub-sections VI.2 and VI.3 solve the disturbance attenuation problem (92).

Proof of Theorem VI.1:

The proof here is brief since it is similar to Theorem V.1. To begin, it is clear that $(w_i, v_i, x_i, u_i, z) = 0$, for $i = 0, 1, 2, \cdots$, satisfy the first order necessary conditions as derived in section VI, therefore zero is indeed the minimax trajectory produced by the minimax strategy. Thus, if Assumptions (4) and (5) are satisfied, then

$$\bar{J}^M = 0 \tag{161}$$

Consequently, the strategies $u_i^*, x_i^m(0), w_i^m, v_i^m,\ \ i = 0, 1, 2, \cdots$, are indeed the minimax controller asymptotically. If the adversaries do not play their minimax strategies,

$$\bar{J}_*(u_0^*, u_1^*, \cdots; x_0(0), x_1(0), \cdots; w_0, w_1, \cdots; v_0, v_1, \cdots) \leq 0 \tag{162}$$

Alternatively, (162) is recast as

$$\bar{D}_{af}(u^*, x(0), w, v) \leq \bar{\theta} \tag{163}$$

for all $w \epsilon L_2[0,T]$, $v \epsilon L_2[0,T]$ and $x(0) \epsilon R^n$ s.t. $(w(\tau), v(\tau)) \neq 0$ for all $\tau \epsilon [0,T]$ and $x(0) \neq \hat{x}(0)$.

$\diamondsuit$

To show that a finite order approximate controller, truncated from the infinite order controller, achieves the disturbance attenuation property, first, $\bar{D}_{af}(u^*, v, w, x(0))$ is decomposed. To proceed, denote

$$u^* = {}^N u^* + {}^R u^* = \sum_{i=0}^{N} u_i^* \epsilon^i + \sum_{i=N+1}^{\infty} u_i^* \epsilon^i. \tag{164}$$

Using (164), $\bar{D}_{af}(u^*, v, w, x(0))$ is rewritten as

$$\bar{D}_{af}(u^*, v, w, x(0)) \equiv \frac{\bar{a} + \bar{b}}{\bar{c}} \leq \bar{\theta}, \tag{165}$$

where

$$\bar{a} \equiv \|x(T)\|^2_{Q_T} + \int_0^T \|x\|^2_Q + \|^N u^*\|^2_R d\tau \tag{166}$$

$$\bar{b} \equiv \int_0^T 2(^N u^*)^T R^R u^* \|^R u^*\|^2_R d\tau \tag{167}$$

$$\bar{c} \equiv \|x(0) - \hat{x}(0)\|^2_{P_0^{-1}} + \int_0^T \|w\|^2_{W^{-1}} + \|v\|^2_{V^{-1}} d\tau \tag{168}$$

Corresponding the disturbance attenuation problem defined in section VI, a N-th order disturbance attenuation problem is defined as to find $^N u^*$ s.t.

$$\bar{D}_{af}(^N u^*, v, w, x(0)) \leq \bar{\theta}' > \bar{\theta} > 0, \tag{169}$$

for all $w, v \epsilon L_2[0, T]$ and $x(0) \epsilon R^n$ such that $(w(\tau), v(\tau)) \neq 0$ for all $\tau \epsilon [0, T]$ and $x(0) \neq \hat{x}(0)$, simultaneously. Clearly

$$\bar{D}_{af}(^N u^*, v, w, x(0)) = \frac{\bar{a}}{\bar{c}} \tag{170}$$

Theorem VI.2:

If the infinite order disturbance attenuation problem as posed in (92) is solved, the N-th order disturbance attenuation problem is achieved with a threshold which is at most twice of the original threshold.

Proof of Theorem VI.2:

From (165) and (170),

$$\bar{D}_{af}(^N u^*, v, w, x(0)) \leq \bar{\theta} + \frac{|\bar{b}|}{\bar{c}} \leq \bar{\theta} + \frac{\bar{a} + \bar{b}}{\bar{c}} \leq 2\bar{\theta} \tag{171}$$

$\diamond$

Theorem VI.3:

For the dynamical system as given in (89) and (90), suppose the infinite-order disturbance attenuation problem as defined in (92) is solved by the infinite-order minimax controller, then the N-th order disturbance attenuation inequality as defined in (169) is achieved by $^N u^*$. The upper bound of the threshold is shown to be proportional to the (N+1)-th power of ϵ.

Proof of Theorem VI.3:

From (167), there exists $0 < K_{c1} < \infty$ such that

$$|\bar{b}| = \int_0^T |2(\sum_{i=0}^{N} u_i^* \epsilon^i)^T R(\sum_{N+1}^{\infty} u_i^* \epsilon^i) + \| \sum_{N+1}^{\infty} u_i^* \epsilon^i \|_R^2 | d\tau \tag{172}$$

$$< K_{c1} \epsilon^{N+1}. \tag{173}$$

From (168), $\bar{c}$ is a zeroth-order term. Moreover, from Theorem VI.1, $\frac{|\bar{b}|}{\bar{c}} \leq \bar{\theta}$. Therefore, there exists K_{c2} satisfying $\frac{K_{c1}}{\bar{c}} \leq K_{c2}$. Thus,

$$\frac{|\bar{b}|}{\bar{c}} < \frac{K_{c1}\epsilon^{N+1}}{\bar{c}} \leq K_{c2}\epsilon^{N+1} \leq \bar{\theta}. \tag{174}$$

As $N \to \infty$, $K_{c2}\epsilon^{N+1} \to 0$, thus $\frac{|\bar{b}|}{\bar{c}} \to 0$. Therefore,

$$\bar{D}_{af}({}^N u^*, v, w, x(0)) = \frac{\bar{a}}{\bar{c}} \leq \bar{\theta} + \frac{|\bar{b}|}{\bar{c}} \to \bar{\theta}. \tag{175}$$

$\diamondsuit$

VII. Conclusion

Both nonlinear minimax estimator and controller are derived via a regular perturbation technique by solving disturbance attenuation problems. The disturbance attenuation problems are first converted to their associated deterministic game problems. Then, adopting a calculus of variation approach, the estimation and control game problem are solved. In the solution processes, both nonlinear two-point boundary value problems and the cost functions are decomposed. Following the perturbation techniques, the expansion terms for the estimator and the output feedback controller are derived sequentially. The optimization of the odd expansion terms of the cost function are proved to be insignificant. Most importantly, using only finite terms, say the zeroth-order and the first order correction term, both the nonlinear approximate estimator and controler are proved to have disturbance attenuation property. In addition, these property is a priori. Using arbitrary term in the estimator and controller, the disturbance attenuation threshold is first proved to be twice of the original optimal threshold. As more terms are used, the threshold decreases according to ϵ^{N+1} and converges to the original one when infinite order is used. This approach is demonstrated here as a very powerful tool to provide implementable estimation scheme and control algorithm in the output feedback case for the class of linear dynamical systems perturbed by a small scale of nonlinearities both in the system dynamics and the measurement process.

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